

On the reduction modulo p of representations of a quaternion division algebra over a p -adic field

Kazuki Tokimoto

1 Introduction

Let F be a non-Archimedean local field with a finite residue field of characteristic p . The local Langlands correspondence (LLC) establishes a canonical bijection between the set of isomorphism classes of n -dimensional Frobenius semisimple Deligne representations of the Weil group \mathcal{W}_F and the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. Recently two kinds of analogues of the correspondence have been studied extensively, that is, the p -adic Langlands correspondence and the mod p Langlands correspondence. The former seeks for an extended correspondence involving p -adic Galois representations (i.e. continuous representations of the absolute Galois group G_F of F over a p -adic field) whereas the latter looks for a correspondence between mod p representations (i.e. representations over a field of characteristic p) and the compatibility of these two correspondences with the reduction modulo p is one of the keys to their studies.

When $n = 2$ and $F = \mathbb{Q}_p$, two correspondences, along with a number of satisfactory properties including the compatibility with the reduction in many cases, have been established (cf. [Br3]). To motivate our study in this paper let us look at an early version of the correspondences. Breuil (cf. [Br1], [Br2]) conjectured and partly proved a p -adic Langlands correspondence involving two-dimensional irreducible crystalline representations of $G_{\mathbb{Q}_p}$ and a mod p Langlands correspondence involving two-dimensional semisimple mod p representations of $G_{\mathbb{Q}_p}$. Roughly speaking, the p -adic Langlands correspondence was proposed by combining the local Langlands correspondence with the data necessary to obtain a p -adic Galois representation from a Deligne representation and the mod p Langlands correspondence was discovered by classifying the mod p representations of both sides by the same parameters and taking into account the compatibility with the reduction. Our objective in this paper is an elementary consideration of a similar situation with $\mathrm{GL}_2(F)$ replaced by the multiplicative group D^\times of a quaternion division algebra D over F .

The local Jacquet-Langlands correspondence (LJLC) gives a canonical bijection between the set of isomorphism classes of discrete series represen-

tations of $\mathrm{GL}_2(F)$ and the set $\mathcal{A}_1(D)$ of isomorphism classes of irreducible admissible representations of D^\times . Composing with LLC, we obtain a canonical bijection between the set of isomorphism classes of two-dimensional indecomposable Deligne representations of \mathcal{W}_F and the set $\mathcal{A}_1(D)$. In this paper we first classify certain irreducible mod p representations of \mathcal{W}_F and of D^\times , thereby proposing a mod p correspondence (cf. section 2), and then compute (the semisimplification of) the reduction of certain representations of \mathcal{W}_F and of D^\times (cf. section 3) to examine if this correspondence is compatible with the composite of LLC and LJLC (cf. section 4).

Irreducible representations of D^\times are easier to deal with than those of $\mathrm{GL}_2(F)$ in many aspects. Since D^\times is compact modulo center, representations we treat are all finite-dimensional. Also, as the pro- p -subgroup U_D^1 is normal not only in the unit group U_D , but also in D^\times , every irreducible mod p representation of D^\times is inflated from that of D^\times/U_D^1 , which makes irreducible mod p representations of D^\times much more accessible than those of $\mathrm{GL}_2(F)$. It follows that the groups essentially involved are isomorphic and a mod p correspondence is obtained in quite a natural manner. However, it turned out that this mod p correspondence and the composed correspondence are not compatible with the reduction modulo p except for the simplest case of “level zero” (the mod p correspondence here and its compatibility with the reduction in the level zero case have already been treated by [Vi2]). In fact, in most cases the reduction of an irreducible representation of D^\times contains every irreducible mod p representation satisfying the obvious necessary condition (i.e. having the suitable central character). This may be natural in that every irreducible mod p representation of D^\times has a non-zero vector fixed by U_D^1 and the same condition forces an irreducible admissible representation of D^\times to be of level zero. At least we make some observation on this phenomenon (cf. 4.5).

Acknowledgements

The author wishes to thank Christophe Breuil for listening to the author at the poster session of a conference and showing a paper [BD] in preparation at the time whose appendix contains a similar computation and Yoichi Mieda for pointing out a silly misunderstanding of the author. Also he would like to express his sincere gratitude to his family for always encouraging him, to his friends for making him feel relaxed and motivating him, and last but not least to his advisor Takeshi Tsuji for helpful advice, illuminating discussions and uplifting words.

The author was supported by the Program for Leading Graduate Schools, MEXT, Japan.

Notation and terminology

We list here the notation and the terminology which we use freely without recalling the definitions. For more on the materials considered here, see [BH], [Se2] and [Re] among many other standard references.

Let F be a non-Archimedean local field with a finite residue field k_F of characteristic p , over which we work in the entire paper. Let q be the cardinality of k_F . Let D denote a quaternion division algebra (i.e. a central division algebra of dimension four) over F .

If E/F is a finite field extension, we denote the normalized discrete valuation of E by $v_E : E^\times \rightarrow \mathbb{Z}$, the valuation ring of E by \mathfrak{o}_E , the maximal ideal of \mathfrak{o}_E by \mathfrak{p}_E and the residue field by k_E . The unit group is denoted by $U_E = \mathfrak{o}_E^\times$ and its congruence subgroups by $U_E^n = 1 + \mathfrak{p}_E^n$ for $n \geq 1$. We denote by μ_E the group of roots of unity of order prime to p in E^\times (canonically isomorphic to k_E^\times). When we take a uniformizer of E , it is usually denoted by ϖ_E . The ramification index of the extension E/F is denoted by $e(E|F)$.

We use an analogous notation for the division algebra D . Let us denote the reduced trace and the reduced norm of D by $\text{Trd} : D \rightarrow F$ and $\text{Nrd} : D^\times \rightarrow F^\times$. We set $v_D = v_F \circ \text{Nrd} : D^\times \rightarrow \mathbb{Z}$. It is a surjective homomorphism. Let $\mathfrak{o}_D = \{x \in D^\times \mid v_D(x) \geq 0\} \cup \{0\}$ and $\mathfrak{q} = \{x \in D^\times \mid v_D(x) \geq 1\} \cup \{0\}$ denote the maximal order of D and its maximal two-sided ideal. The residue ring $k_D = \mathfrak{o}_D/\mathfrak{q}$ is a finite field with q^2 elements. We set $U_D = \mathfrak{o}_D^\times = \text{Ker } v_D$ and $U_D^n = 1 + \mathfrak{q}^n$ for $n \geq 1$. If $\varpi_D \in D^\times$ satisfies $v_D(\varpi_D) = 1$, it acts on k_D by conjugation as a q -th power (i.e. $\varpi_D x \varpi_D^{-1} \equiv x^q \pmod{U_D^1}$ for all $x \in U_D$).

If E/F is a finite extension, \mathcal{W}_E denotes the Weil group of E . If we take a Frobenius element $\text{Fr} \in \mathcal{W}_E$, then \mathcal{W}_E is the semidirect product $\mathcal{I}_E \rtimes \text{Fr}^\mathbb{Z}$ of the inertia subgroup \mathcal{I}_E and the discrete infinite cyclic subgroup $\text{Fr}^\mathbb{Z}$. The unique pro- p -Sylow subgroup \mathcal{P}_E of \mathcal{I}_E is called the wild inertia subgroup and the tame inertia group $\mathcal{I}_E/\mathcal{P}_E$ is canonically isomorphic to $\varprojlim_n \mathbb{F}_{p^n}^\times$, where \mathbb{F}_{p^n} denotes the field with p^n elements. Through this isomorphism the conjugation of the tame inertia group by a Frobenius element Fr amounts to the usual action of Fr on $\varprojlim_n \mathbb{F}_{p^n}^\times$ (i.e. if $i : \mathcal{I}_E/\mathcal{P}_E \rightarrow \varprojlim_n \mathbb{F}_{p^n}^\times$ stands for the canonical isomorphism, we have $i((\text{Fr})g(\text{Fr})^{-1}) = \text{Fr}(i(g))$). The local class field theory provides a natural homomorphism $\mathbf{a}_E : \mathcal{W}_E \rightarrow E^\times$ (Artin reciprocity map) which induces an isomorphism between the abelianization $\mathcal{W}_E^{\text{ab}}$ and E^\times . We normalize it so that geometric Frobenius elements are sent to uniformizers.

From now on, we fix a field isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ and the representations over \mathbb{C} are invariably identified with the representations over $\overline{\mathbb{Q}_p}$ through this isomorphism. In this paper, we consider admissible representations of locally profinite groups over \mathbb{C} (or $\overline{\mathbb{Q}_p}$) and over $\overline{\mathbb{F}_p}$. To specify it the latter will often be called “mod p representations” although it will be usually clear from the context which we are dealing with. One-dimensional smooth representations

(i.e. continuous homomorphisms to \mathbb{C}^\times or $\overline{\mathbb{F}}_p^\times$) are often called characters.

Let E/F be a finite extension. If ψ is a character of E (resp. of D), the level of ψ is defined to be the least integer n such that $\mathfrak{p}_E^n \subset \text{Ker } \psi$ (resp. $\mathfrak{q}^n \subset \text{Ker } \psi$). If χ is a character of E^\times , the level of χ is defined to be the least integer n such that $U_E^{n+1} \subset \text{Ker } \chi$. Similarly, if Π is an irreducible representation of D^\times of dimension greater than one, the level of Π is defined to be the least integer n such that $U_D^{n+1} \subset \text{Ker } \Pi$.

If G is a locally profinite group and H is a closed subgroup, $\text{Ind}_H^G \sigma$ denotes the smooth induction of a smooth representation σ of H to G . (However, H is always of finite index in G in this paper and hence the notion of smooth induction and that of compact induction coincide.) Also, $\text{Res}_H^G \sigma$ and $\sigma|_H$ denotes the restricted representation of a smooth representation σ of G to H .

2 Classification of mod p representations

2.1 Irreducible mod p representations of D^\times

In this subsection we classify irreducible admissible mod p representations of D^\times . As the abelianization $D^\times \rightarrow (D^\times)^{\text{ab}}$ is isomorphic to the reduced norm map $D^\times \xrightarrow{\text{Nrd}} F^\times$, mod p characters of D^\times are in a natural one-to-one correspondence with mod p characters of F^\times (by composing $\text{Nrd} : D^\times \rightarrow F^\times$). We denote the set of isomorphism classes of irreducible admissible representations of D^\times over $\overline{\mathbb{F}}_p$ of dimension greater than one by $\mathcal{A}_1^0(D, \overline{\mathbb{F}}_p)$.

We begin with introducing the language which we use throughout the paper.

Definition 2.1. *Let K be an algebraically closed field.*

An admissible pair $(E/F, \chi)$ with values in K is a pair in which E/F is a tamely ramified quadratic extension and χ is a character of E^\times with values in K , satisfying the following conditions:

1. χ does not factor through the norm map $\text{N}_{E/F} : E^\times \rightarrow F^\times$ (i.e. there does not exist any character φ of F^\times such that $\chi = \varphi \circ \text{N}_{E/F}$).
2. if $\chi|_{U_E^1}$ does factor through the norm map $\text{N}_{E/F}$, then E is unramified over \tilde{F} .

The level of an admissible pair $(E/F, \chi)$ is defined to be the level of χ . An admissible pair $(E/F, \chi)$ is said to be minimal if the level of $(\varphi \circ \text{N}_{E/F}) \otimes \chi$ is not strictly smaller than that of χ for any character φ of F^\times . We often call minimal admissible pairs simply by minimal pairs.

We say that admissible pairs $(E/F, \chi)$ and $(E'/F, \chi')$ are F -isomorphic if there exists an F -isomorphism $j : E \rightarrow E'$ such that $\chi = \chi' \circ j$. The set of F -isomorphism classes of admissible pairs with values in K is denoted by

$\mathbb{P}_2(F, K)$. If $K = \mathbb{C}$, an admissible pair with values in \mathbb{C} is simply called an admissible pair and we set $\mathbb{P}_2(F) = \mathbb{P}_2(F, \mathbb{C})$.

Remark 2.2. 1. Since every character of a subgroup of F^\times of index two with values in K can be extended to a character of F^\times , Hilbert 90 shows that the condition 1 in the definition is equivalent to

$$\chi \neq \chi^\theta \text{ for } \theta \in \text{Gal}(E/F) \setminus \{1\}.$$

2. Admissible pairs $(E/F, \chi)$ and $(E/F, \chi')$ are F -isomorphic if and only if $\chi^\theta = \chi'$ for some element $\theta \in \text{Gal}(E/F)$.
3. If χ in an admissible pair $(E/F, \chi)$ is trivial on U_E^1 (e.g. if the characteristic of K is p), E is unramified over F by the condition 2.

We also need another closely related notion to facilitate the computation of the reduction.

Definition 2.3. Let E/F be a quadratic Galois extension. Let ξ be a mod p character of E^\times .

We say that ξ is regular if ξ does not factor through the norm map $N_{E/F} : E^\times \rightarrow F^\times$.

Remark 2.4. 1. If E/F is an unramified quadratic extension, we have a natural identification

$$\mathbb{P}_2(F, \overline{\mathbb{F}}_p) = \{\xi \mid \xi \text{ is a regular mod } p \text{ character of } E^\times\} / \sim,$$

where \sim is an equivalence relation defined by

$$\xi \sim \xi' \text{ if and only if } \xi^\theta = \xi'$$

for some element $\theta \in \text{Gal}(E/F)$.

2. Let E/F and ξ be as in the Definition 2.3.

If E/F is unramified, then ξ is regular if and only if $\xi(\zeta_E^{q-1}) \neq 1$, where $\zeta_E \in E^\times$ is a generator of μ_E .

If E/F is totally ramified, then ξ is regular if and only if $\xi(-1) \neq 1$. (Note that ξ is trivial on U_E^1 .) In particular, ξ is irregular if $p = 2$.

3. As in the previous remark, we often take some elements to state the result explicitly. Therefore we fix the notation here to avoid explaining the same setting repetitively.

- (a) If a quadratic unramified extension E/F (sometimes denoted by E_0/F) is fixed in the context, then we take a generator $\zeta_E \in \mu_E$ and a uniformizer $\varpi_F \in F$, and set $\zeta_F = \zeta_E^{q+1} \in \mu_F$.

- (b) If a tamely quadratic totally ramified extension E/F is fixed in the context, then we take a generator $\zeta_F \in \mu_F$ and a uniformizer $\varpi_E \in E$ such that ϖ_E^2 is a uniformizer of F , and set $\varpi_F = \varpi_E^2 \in F$.

In what follows, we mainly use the more flexible notion of regular mod p characters instead of that of admissible pairs with values in $\overline{\mathbb{F}}_p$.

Proposition 2.5. *Let E/F be an unramified quadratic extension and ξ a mod p character of E^\times . Let us take an F -embedding $E \rightarrow D$. Let us denote by ξ^\dagger the mod p character of $E^\times U_D^1 = F^\times U_D$ such that $\xi^\dagger|_{E^\times} = \xi$ and $\xi^\dagger|_{U_D^1} = 1$. Finally, let us set*

$$\pi_\xi = \text{Ind}_{E^\times U_D^1}^{D^\times} \xi^\dagger.$$

Then the following assertions hold.

1. The representation π_ξ is independent up to isomorphism of the choice of F -embedding $E \rightarrow D$.
2. The representation π_ξ is irreducible if and only if ξ is regular. Every irreducible mod p representation of D^\times of dimension greater than one can be expressed this way. If ξ and ξ' are regular mod p characters, π_ξ and $\pi_{\xi'}$ are isomorphic if and only if $\xi^\theta = \xi'$ for some $\theta \in \text{Gal}(E/F)$.

In other words, the construction above induces a bijection

$$\mathbb{P}_2(F, \overline{\mathbb{F}}_p) \simeq \mathcal{A}_1^0(D, \overline{\mathbb{F}}_p). \quad (1)$$

In particular, irreducible admissible mod p representations of D^\times are either one-dimensional or two-dimensional.

3. Let ξ be irregular. Then ξ^\dagger admits an extension to a mod p character of D^\times . There exist two such extensions $\tilde{\xi}_1, \tilde{\xi}_2$ if $p \neq 2$ and only one extension $\tilde{\xi}$ if $p = 2$. We have

$$\begin{aligned} \pi_\xi &\cong \tilde{\xi}_1 \oplus \tilde{\xi}_2 && \text{if } p \neq 2, \\ \pi_\xi &\text{ is a non-split extension of } \tilde{\xi} \text{ with itself} && \text{if } p = 2. \end{aligned}$$

Proof. Changing the choice of F -embedding only amounts to the composition of an inner automorphism of D^\times by Skolem-Noether theorem and hence 1 follows immediately.

Since π_ξ is two-dimensional, it is reducible if and only if it has a one-dimensional subrepresentation, which is equivalent to the existence of an extension of ξ^\dagger to D^\times by Frobenius reciprocity. One can verify by an elementary calculation that ξ^\dagger extends to D^\times if and only if ξ^\dagger is fixed under the

conjugation action of $\varpi_D^{\mathbb{Z}}$, where $\varpi_D \in D^\times$ satisfies $v_D(\varpi_D) = 1$. Observing that the mod p character ξ^\dagger is trivial on the normal pro- p -subgroup U_D^1 and $\varpi_D \zeta_E \varpi_D^{-1} \equiv \zeta_E^q \pmod{U_D^1}$ for any generator $\zeta_E \in \mu_E$, we see that this is certainly equivalent to ξ being irregular. If ξ is irregular, any extensions of ξ^\dagger appear both as a subrepresentation and as a quotient representation of π_ξ .

If $p \neq 2$ and ξ is irregular, π_ξ has two distinct subrepresentations $\tilde{\xi}_1$ and $\tilde{\xi}_2$ and is therefore a direct sum of these subrepresentations.

Mackey's decomposition [Se1, 7.3], [Vi1, I.5.5] gives $\text{Res}_{E^\times U_D^1}^{D^\times} \pi_\xi \cong \xi^\dagger \oplus (\xi^\theta)^\dagger$ and Frobenius reciprocity yields

$$\text{Hom}_{D^\times}(\pi_{\xi'}, \pi_\xi) \cong \text{Hom}_{E^\times U_D^1}(\xi'^\dagger, \xi^\dagger \oplus (\xi^\theta)^\dagger),$$

for any mod p characters ξ and ξ' of E^\times . If ξ is regular, this implies the stated condition for $\pi_\xi \cong \pi_{\xi'}$. Taking $\xi = \xi'$ to be irregular, we see that $\text{End}_{D^\times}(\pi_\xi)$ is two-dimensional and hence π_ξ is not a direct sum of two isomorphic representations if $p = 2$ and ξ is irregular.

It now remains to show that the induced map (1) is surjective. Let (π, V) be an irreducible mod p representation of D^\times of dimension greater than one. As U_D^1 is pro- p and V is a direct limit of p -groups, a standard argument shows that the invariant part $V^{U_D^1} \neq 0$.¹ Since the order of U_D/U_D^1 is prime to p , $V^{U_D^1}$ is semisimple as a representation of U_D and in particular contains a character of U_D . On the other hand, π admits a central character by Schur's lemma. These characters are consistent and define a character ξ' of $F^\times U_D = E^\times U_D^1$ contained in V . Let us put $\xi = \xi'|_E$. By Frobenius reciprocity the inclusion of ξ' into V induces a D^\times -equivariant map $\pi_\xi = \text{Ind}_{F^\times U_D}^{D^\times} \xi' \rightarrow \pi$, which is non-zero and hence surjective by irreducibility. Since π is not one-dimensional, it is an isomorphism. This completes the proof. \square

Remark 2.6. *If ξ is irregular (and E/F is unramified), then any extension of ξ is expressed as $(\varphi \circ \text{Nrd})$ with some mod p character φ of F^\times such that*

$$\varphi(\zeta_F) = \xi(\zeta_E) \text{ and } \varphi(\varpi_F)^2 = \xi(\varpi_F),$$

where ζ_E, ζ_F and ϖ_F are taken as in Remark 2.4.3a.

Although this much suffices for the classification of irreducible mod p representations of D^\times , we study further the analogous situation with E being ramified over F for the convenience of computations later.

¹In fact, U_D^1 is normal in D^\times and hence $V^{U_D^1} = V$ by irreducibility. In other words, every irreducible mod p representation of D^\times is inflated from that of D^\times/U_D^1 . However, we are not going to need this fact in what follows.

Proposition 2.7. *Let E/F be a totally ramified quadratic extension of F and ν a mod p character of E^\times . Let us take an F -embedding $E \rightarrow D$. Let us denote by ν^\dagger the extension of ν to $E^\times U_D^1$ such that $\nu^\dagger|_{U_D^1} = 1$.*

Then the induced representation $\text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger$ is independent up to isomorphism of the choice of F -embedding and we have

$$\text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger \cong \left(\bigoplus_{\pi \in I_1} \pi \right) \oplus \left(\bigoplus_{\tilde{\nu} \in I_2} \tilde{\nu} \right),$$

where I_1 denotes the set of (isomorphism classes of) two-dimensional irreducible mod p representations of D^\times with a central character $\nu|_{F^\times}$ and I_2 denotes the set of mod p characters of D^\times extending ν^\dagger .

If E_0 denotes an unramified quadratic extension of F , we have

$$I_1 = \{ \pi_\xi \mid \xi \text{ is a regular mod } p \text{ character of } E_0^\times \text{ extending } \nu|_{F^\times} \},$$

$$I_2 = \{ \varphi \circ \text{Nrd} \mid \varphi \text{ is a mod } p \text{ character of } F^\times \text{ such that } (\varphi \circ \text{Nrd})|_{E^\times} = \nu \},$$

and

$$(\#I_1, \#I_2) = \begin{cases} (\frac{q+1}{2}, 0) & \text{if } \nu \text{ is regular} \\ (\frac{q-1}{2}, 2) & \text{if } \nu(-1) = 1 \text{ and } p \neq 2 \\ (\frac{q}{2}, 1) & \text{if } p = 2. \end{cases}$$

Proof. It can be seen in exactly the same way as in the preceding proposition that the induced representation is independent up to isomorphism of the choice of F -embedding.

Let us fix an F -embedding $E_0 \rightarrow D$. Mackey's decomposition gives

$$\text{Res}_{E_0^\times U_D^1}^{D^\times} \text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger \cong \text{Ind}_{F^\times U_D^1}^{E_0^\times U_D^1} (\nu^\dagger|_{F^\times U_D^1}).$$

Frobenius reciprocity then induces non-zero D^\times -equivariant maps $\pi_\xi \rightarrow \text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger$ for any mod p characters ξ of E_0^\times extending $\nu|_{F^\times}$. Of all such extensions ξ , regular characters come in pairs and give rise to isomorphic two-dimensional irreducible representations.

Let us take $\zeta_{E_0} \in \mu_{E_0}$ and $\zeta_F \in \mu_F$ as in Remark 2.4.3a. If a mod p character ξ of E_0^\times is an extension of $\nu|_{F^\times}$, we have $\xi(\zeta_{E_0})^{q+1} = \nu(\zeta_F)$, whereas ξ is irregular if and only if $\xi(\zeta_{E_0}^q) = \xi(\zeta_{E_0})$. It follows that $\nu|_{F^\times}$ admits an irregular extension to E_0^\times if and only if $\nu(-1) = 1$ (i.e. ν is irregular) and if there are any such extensions

$$\begin{aligned} & \text{there are two} && \text{if } p \neq 2, \\ & \text{and there is only one} && \text{if } p = 2. \end{aligned}$$

For such an irregular character ξ of E_0^\times , the character ξ^\dagger of $E_0^\times U_D^1$ extends to at most two mod p characters of D^\times (two if $p \neq 2$ and one if $p = 2$). Among these characters, only the character extending ν^\dagger occurs in $\text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger$.

Summing up dimensions, we obtain the required decomposition of the induced representation $\text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger$ and the other assertions now follow easily. \square

Remark 2.8. Let E , E_0 , ν and F -embeddings $E \rightarrow D$, $E_0 \rightarrow D$ be as above.

1. For a mod p character ξ of E_0^\times to be an extension of $\nu|_{F^\times}$, it is necessary and sufficient that

$$\xi(\zeta_{E_0})^{q+1} = \nu(\zeta_F) \text{ and } \xi(\varpi_F) = \nu(\varpi_F)$$

and an extension ξ is regular if and only if

$$\xi(\zeta_{E_0})^2 \neq \nu(\zeta_F).$$

with ζ_{E_0}, ζ_F and ϖ_F as in Remark 2.4.3a.

2. If ν is irregular, then any extension of ν to D^\times is expressed as $(\varphi \circ \text{Nrd})$ with some mod p character φ of F^\times . If $p \neq 2$, then φ satisfies

$$\varphi(\zeta_F)^2 = \nu(\zeta_F) \text{ and } \varphi(-\varpi_F) = \nu(\varpi_F).$$

with ζ_F, ϖ_E and ϖ_F as in Remark 2.4.3b, and if $p = 2$, then φ is the unique mod p character such that $\varphi^2 = \nu|_{F^\times}$.

2.2 Two-dimensional semisimple mod p representations of \mathcal{W}_F

In order to establish a mod p correspondence, we classify two-dimensional semisimple smooth mod p representations of \mathcal{W}_F in this subsection. Very much analogous to what holds in D^\times , one-dimensional mod p representations of \mathcal{W}_F are parametrized by mod p characters of F^\times by simply composing the Artin reciprocity map $\mathbf{a}_F : \mathcal{W}_F \rightarrow F^\times$. We denote the set of isomorphism classes of two-dimensional irreducible smooth representations of \mathcal{W}_F over $\overline{\mathbb{F}}_p$ by $\mathcal{G}_2^0(F, \overline{\mathbb{F}}_p)$.

We are going to take a look at two-dimensional induced mod p representations of \mathcal{W}_F and in particular parametrize elements of $\mathcal{G}_2^0(F, \overline{\mathbb{F}}_p)$. For the most part, the proofs are quite similar to Proposition 2.5. However, observe the differences in the assertions when E/F is ramified.

Proposition 2.9. Let E/F be a quadratic Galois extension of F and ξ a mod p character of E^\times .

1. The induced representation $\rho_\xi = \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\xi \circ \mathbf{a}_E)$ is irreducible if and only if ξ is regular. If ξ and ξ' are regular mod p characters of E^\times , then $\rho_\xi \cong \rho_{\xi'}$ holds if and only if $\xi^\theta = \xi'$ for some $\theta \in \text{Gal}(E/F)$.

Any two-dimensional irreducible mod p representations of \mathcal{W}_F arise this way with E/F unramified.

In particular, this construction induces a bijection

$$\mathbb{P}_2(F, \overline{\mathbb{F}}_p) \simeq \mathcal{G}_2^0(F, \overline{\mathbb{F}}_p). \quad (2)$$

2. Let ξ be irregular. Then $(\xi \circ \mathbf{a}_E)$ admits an extension to a mod p character of \mathcal{W}_F . There are two such extensions λ_1, λ_2 if $p \neq 2$ and only one extension λ if $p = 2$. We have

$$\begin{aligned} \rho_\xi &\cong \lambda_1 \oplus \lambda_2 && \text{if } p \neq 2, \\ \rho_\xi &\text{ is a non-split extension of } \lambda \text{ with itself} && \text{if } p = 2. \end{aligned}$$

Proof. As the induced representation ρ_ξ is two-dimensional, it is reducible if and only if it has a one-dimensional subrepresentation, which is then, by Frobenius reciprocity and the functoriality of the Artin reciprocity map, equivalent to ξ factoring through $N_{E/F} : E^\times \rightarrow F^\times$, i.e. being irregular.

If ρ_ξ does contain one-dimensional subrepresentation, $(\xi \circ \mathbf{a}_E)$ admits two extensions λ_1, λ_2 to a mod p character of \mathcal{W}_F if $p \neq 2$ and only one extension λ if $p = 2$. In any case, Frobenius reciprocity shows that ρ_ξ has every extension of $(\xi \circ \mathbf{a}_E)$ as a subrepresentation and a quotient representation.

If $p \neq 2$, the two-dimensional representation ρ_ξ contains two distinct characters λ_1, λ_2 and hence is isomorphic to the direct sum $\lambda_1 \oplus \lambda_2$.

Frobenius reciprocity yields

$$\mathrm{Hom}_{\mathcal{W}_F}(\rho_{\xi'}, \rho_\xi) \cong \mathrm{Hom}_{\mathcal{W}_E}(\xi' \circ \mathbf{a}_E, (\xi \circ \mathbf{a}_E) \oplus (\xi^\theta \circ \mathbf{a}_E)).$$

Taking ξ and ξ' to be regular, we obtain the required condition for $\rho_\xi \cong \rho_{\xi'}$ when ξ is regular. Taking $\xi = \xi'$ to be irregular, we see that $\mathrm{End}_{\mathcal{W}_F}(\rho_\xi)$ is two-dimensional and hence ρ_ξ is a non-split extension if $p = 2$ and ξ is irregular.

Now it only remains to show that the induced map (2) is surjective. Let (ρ, V) be a two-dimensional irreducible mod p representation of \mathcal{W}_F and let E_0/F be an unramified quadratic extension. As the wild inertia subgroup \mathcal{P}_F is a normal pro- p -subgroup of \mathcal{W}_F , it acts trivially on V .² Since the tame inertia group $\mathcal{I}_F/\mathcal{P}_F$ is abelian, profinite and of pro-order prime to p , the restricted representation $\rho|_{\mathcal{I}_F}$ is a direct sum of two mod p characters. These two characters are distinct. Indeed, if they were identical, \mathcal{I}_F would act on V as scalar operators and an eigenspace of the action of a Frobenius element would give a proper subrepresentation, which is a contradiction.

²Thus any irreducible mod p representation of \mathcal{W}_F is inflated from a representation of the semidirect product of the tame inertia group $\mathcal{I}_F/\mathcal{P}_F$ and an infinite cyclic group generated by a Frobenius element. Irreducible representations of a finite group which is a semidirect product with the normal subgroup abelian are well-understood. See, for instance, [Se1, 8.2]. The argument here can be considered as a modification.

Let $V = V_1 \oplus V_2$ be the decomposition of $\rho|_{\mathcal{I}_F}$ into irreducible representations. Let us take a Frobenius element $\text{Fr} \in \mathcal{W}_F$. The subgroup $(\text{Fr})^{\mathbb{Z}}$ permutes the set $\{V_1, V_2\}$ transitively. The stabilizer of V_1 with respect to this action is $(\text{Fr})^{2\mathbb{Z}}$ and V_1 defines a mod p character of $\mathcal{I}_F(\text{Fr})^{2\mathbb{Z}} = \mathcal{W}_{E_0}$. Expressing this character as $(\xi \circ \mathbf{a}_{E_0})$ with a suitable character ξ of E_0^\times , we obtain $\rho \cong \rho_\xi$. \square

Remark 2.10. Let E_0/F be an unramified quadratic extension and E/F a totally ramified quadratic extension. Let us take a uniformizer $\varpi_E \in E^\times$ so that $\varpi_E^2 \in F^\times$.

1. If ν is a regular mod p character of E^\times (in which case $p \neq 2$), then ρ_ν is isomorphic to ρ_ξ for a mod p character ξ of E_0^\times such that

$$\xi(\zeta_{E_0})^2 = \nu(\zeta_F) \text{ and } \xi(\varpi_F) = \nu(-1)^{(q+1)/2} \nu(\varpi_F).$$

with ζ_{E_0}, ζ_F and ϖ_F as in Remark 2.4.3a.

2. If ξ is an irregular mod p character of E_0^\times , any extension of $(\xi \circ \mathbf{a}_{E_0})$ to \mathcal{W}_F is expressed as $(\varphi \circ \mathbf{a}_F)$ with some mod p character φ of F^\times such that

$$\varphi(\zeta_F) = \xi(\zeta_{E_0}) \text{ and } \varphi(\varpi_F)^2 = \xi(\varpi_F).$$

with ζ_{E_0}, ζ_F and ϖ_F as above.

3. If ν is an irregular mod p character of E^\times , any extension of $(\nu \circ \mathbf{a}_E)$ to \mathcal{W}_F is expressed as $(\varphi \circ \mathbf{a}_F)$ with some mod p character φ of F^\times . If $p \neq 2$, then φ satisfies

$$\varphi(\zeta_F)^2 = \nu(\zeta_F) \text{ and } \varphi(-\varpi_F) = \nu(\varpi_E).$$

with ζ_F, ϖ_E and ϖ_F as in Remark 2.4.3b, and if $p = 2$, then φ is the unique mod p character such that $\varpi^2 = \nu|_{F^\times}$.

2.3 Mod p correspondence

By Propositions 2.9 and 2.5, two-dimensional irreducible mod p representations of \mathcal{W}_F and irreducible mod p representations of D^\times of dimension greater than one are parametrized by the same set $\mathbb{P}_2(F, \overline{\mathbb{F}}_p)$ and therefore naturally correspond to each other. However, motivated by the correspondences in characteristic zero (cf. 4.1), we adjust the correspondence slightly by composing a certain permutation of $\mathbb{P}_2(F, \overline{\mathbb{F}}_p)$.

Definition 2.11. Let E/F be an unramified quadratic extension of F .

We define δ to be the unramified mod p character of E^\times (i.e. δ is trivial on U_E) sending any uniformizer to -1 . (In particular, δ is trivial if $p = 2$.)

It can immediately be seen that the association $\xi \mapsto \delta \otimes \xi$ induces a bijection $\mathbb{P}_2(F, \overline{\mathbb{F}}_p) \rightarrow \mathbb{P}_2(F, \overline{\mathbb{F}}_p)$.

Definition 2.12. We call the bijection between $\mathcal{G}_2^0(F, \overline{\mathbb{F}}_p)$ and $\mathcal{A}_1^0(D, \overline{\mathbb{F}}_p)$ induced by the association $\rho_\xi \mapsto \pi_{\delta \otimes \xi}$ the mod p correspondence.

Remark 2.13. As noted in 1, this correspondence has already been established by Vignéras (cf. [Vi2]).

3 Reduction modulo p of representations of D^\times and of \mathcal{W}_F

3.1 Review of irreducible admissible representations of D^\times

Recall that we fixed an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ in 1. As the topology of the coefficient field is irrelevant to the notion of admissible representations, each irreducible admissible representation of D^\times over \mathbb{C} corresponds to that over $\overline{\mathbb{Q}}_p$ through this isomorphism.

Here we first give a review of the classification of irreducible admissible representations of D^\times over \mathbb{C} , restricting ourselves to the *tame* case in the sense below. (Note that an irreducible admissible representation of D^\times is automatically tamely ramified unless $p = 2$ or it is a character.) These representations are parametrized by admissible pairs with values in \mathbb{C} . All the proofs can essentially be found in [BH] except that they give a complete classification of irreducible admissible representations of both $GL_2(F)$ and D^\times while the parametrization of tame (cuspidal) representations are described only in the case of $GL_2(F)$. Then we briefly recall what we need about the remaining class of irreducible admissible representations. Although our treatment is incomplete in view of the theory of types it suffices for our purposes.

Irreducible admissible representations of D^\times consist of one-dimensional representations, which factor through $\text{Nrd} : D^\times \rightarrow F^\times$, and n -dimensional representations with $1 < n < \infty$. We denote the set of isomorphism classes of the latter representations by $\mathcal{A}_1^0(D)$. Elements in $\mathcal{A}_1^0(D)$ correspond to irreducible cuspidal representations of $GL_2(F)$ via LJLC and then in turn to irreducible two-dimensional representations of \mathcal{W}_F via LLC.

Definition 3.1. Let Π represent an element in $\mathcal{A}_1^0(D)$.

The representation Π is said to be unramified if there exists a non-trivial unramified character Φ of F^\times such that $(\Phi \circ \text{Nrd}) \otimes \Pi \cong \Pi$, and totally ramified otherwise. Also, it is said to be tamely ramified if $p \neq 2$ or it is unramified.

We denote the set of isomorphism classes of unramified representations of D^\times by $\mathcal{A}_1^{nr}(D)$.

We are now going to construct a map $\mathbb{P}_2(F) \rightarrow \mathcal{A}_1^0(D)$.

Let us begin with an admissible pair $(E/F, \chi)$ of level zero. Then E/F is unramified by the definition of admissible pairs. We take an F -embedding $E \rightarrow D$. We extend χ to a character Λ of $J = E^\times U_D^1 (= F^\times U_D)$ by setting $\Lambda|_{U_D^1} = 1$. Then we set

$$\Pi_\chi = \text{Ind}_J^{D^\times} \chi.$$

It is an irreducible admissible representation of D^\times .

Now let $(E/F, \chi)$ be a *minimal* pair of positive level $m > 0$. We fix a character ψ of F of level one. Let us set $\psi_D = \psi \circ \text{Tr}_D$, $\psi_E = \psi \circ \text{tr}_{E/F}$ and $n = 2m/e(E/F)$. The characters ψ_D and ψ_E are both of level one. We take an F -embedding $E \rightarrow D$ and identify E with an F -subalgebra of D . There exists an $\alpha \in \mathfrak{p}_E^{-m}$ such that $\chi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{[m/2]+1}$. (Here, $[l]$ denotes the greatest integer not greater than l .) Then $n = -v_D(\alpha)$. Similarly, we define a character ψ_α of $U_D^{[n/2]+1}$ trivial on U_D^{n+1} by

$$\psi_\alpha(1+x) = \psi_D(\alpha x)$$

for all $x \in \mathfrak{q}^{[n/2]+1}$. We would like to define an element in the set $C(\psi_\alpha)$ of isomorphism classes of irreducible representations Λ of the group $J_\alpha = E^\times U_D^{[(n+1)/2]}$ such that $\Lambda|_{U_D^{[n/2]+1}}$ is a multiple of ψ_α .

First we treat the case where n is odd. Then E/F is totally ramified and $J_\alpha = E^\times U_D^{[n/2]+1}$. In this case we define a character Λ of J_α by

$$\Lambda|_{U_D^{[n/2]+1}} = \psi_\alpha, \quad \Lambda|_{E^\times} = \chi.$$

Then Λ defines a class in $C(\psi_\alpha)$.

Next we move on to the case where $n \equiv 0 \pmod{4}$. This case is almost the same as the previous one, except that the extension E/F is unramified. We have $J_\alpha = E^\times U_D^{[n/2]+1}$ and define a character Λ of J_α by

$$\Lambda|_{U_D^{[n/2]+1}} = \psi_\alpha, \quad \Lambda|_{E^\times} = \chi.$$

Again, Λ defines a class in $C(\psi_\alpha)$.

Finally, if $n \equiv 2 \pmod{4}$, $E^\times U_D^{[n/2]+1}$ has index q^2 in J_α . Let us set $H_\alpha^1 = U_E^1 U_D^{[n/2]+1} \subsetneq J_\alpha^1 = U_E^1 U_D^{[(n+1)/2]}$. Defining

$$\theta(ux) = \chi(u)\psi_\alpha(x)$$

for $u \in U_E^1$ and $x \in U_D^{[n/2]+1}$, we obtain a character θ of H_α^1 .

Proposition 3.2. *1. Under the conditions above there exists an irreducible representation η_θ of J_α^1 containing θ . It is unique up to isomorphism and q -dimensional.*

2. Furthermore, there exists an irreducible representation Λ of J_α such that

- (a) $\Lambda|_{J_\alpha^1} \cong \eta_\theta$
- (b) $\Lambda|_{F^\times} \cong (\chi|_{F^\times})^{\oplus q}$
- (c) $\text{tr } \Lambda(\zeta) = -\chi(\zeta) \quad (\zeta \in \mu_E \setminus \mu_F)$.

It is unique up to isomorphism.

The obtained representation Λ defines a class in $C(\psi_\alpha)$.

Thus we now have an irreducible representation Λ of J_α in all cases. We set

$$\Pi_\chi = \text{Ind}_{J_\alpha}^{D^\times} \Lambda.$$

Now let $(E/F, \chi)$ be a (not necessarily minimal) admissible pair. There exist a minimal pair $(E/F, \chi')$ and a character Φ of F^\times such that $\chi = (\Phi \circ \text{N}_{E/F}) \otimes \chi'$. We define

$$\Pi_\chi = (\Phi \circ \text{Nrd}) \otimes \Pi_{\chi'}.$$

In the course of the definition of Π_χ , we made many choices. Nonetheless, we have

Proposition 3.3. *The isomorphism class of Π_χ only depends on the isomorphism class of an admissible pair $(E/F, \chi)$. Moreover, the representations Π_χ constructed above are irreducible and admissible. In other words, the construction above indeed induces a well-defined map $\mathbb{P}_2(F) \rightarrow \mathcal{A}_1^0(D)$.*

Theorem 3.4. *Sending the isomorphism class $[(E/F, \chi)]$ of an admissible pair $(E/F, \chi)$ to the isomorphism class $[\Pi_\chi]$ of an irreducible admissible representation Π_χ we obtain a bijection Π between the set $\mathbb{P}_2(F)$ of isomorphism classes of admissible pairs and the set of isomorphism classes of tamely ramified representations of D^\times :*

$$\mathbb{P}_2(F) \simeq \mathcal{A}_1^0(D), \quad [(E/F, \chi)] \mapsto [\Pi_\chi] \quad \text{if } p \neq 2$$

$$\mathbb{P}_2(F) \simeq \mathcal{A}_1^{\text{nr}}(D), \quad [(E/F, \chi)] \mapsto [\Pi_\chi] \quad \text{if } p = 2.$$

If $(E/F, \chi)$ is an admissible pair, we have

1. the level of Π_χ is $n = 2m/e(E|F)$,
2. the central character of Π_χ is $\chi|_{F^\times}$,
3. $\Pi_{(\Phi \circ \text{N}_{E/F}) \otimes \chi} \cong (\Phi \circ \text{Nrd}) \otimes \Pi_\chi$ if Φ is a character of F^\times ,
4. and Π_χ is unramified if and only if E/F is unramified.

We only state the corresponding theorem for not tamely ramified representations in a very crude form. The proof of the theorem in a more refined form is found again in [BH].

Theorem 3.5. *Assume that $p = 2$. Let Π represent an element in $\mathcal{A}_1^0(D) \setminus \mathcal{A}_1^{\text{nr}}(D)$. Let $n \in \mathbb{Z}$ be the minimal integer among the levels of representations obtained from Π by a character twist. Then n is odd and there exists a totally ramified quadratic extension E/F embedded in D and a character χ of $E^\times U_D^{(n+1)/2}$ such that $\Pi \cong \text{Ind}_{E^\times U_D^{(n+1)/2}}^{D^\times} \chi$.*

3.2 Reduction of irreducible representations of D^\times

First let us clarify what we mean by reduction. If Π is a representation of finite length, $(\Pi)^{\text{ss}}$ denotes the semisimplification (i.e. the direct sum of composition factors with the appropriate multiplicities) of Π . Every representation (Π, V) of a locally profinite group G over $\overline{\mathbb{Q}_p}$ considered in this paper is finite-dimensional and can be realized over a finite extension K of \mathbb{Q}_p . Let A be the ring of integers in K and \mathfrak{m} the maximal ideal of A . The representation (Π, V) is said to be p -integral if there exists an A -lattice $\Lambda \subset V$ stable under G . If Π is p -integral, the semisimplification of the representation $\Lambda/\mathfrak{m}\Lambda$ over A/\mathfrak{m} is independent up to isomorphism of the choice of a lattice Λ (cf. [Se1, Chap.15.2], [Vi1, Chap1.9]). We consider the semisimplification as a representation over $\overline{\mathbb{F}_p}$, call it, by abuse of language, *the reduction modulo p* (or even more simply, *the reduction*) of Π and denote it by $\overline{\Pi}^{\text{ss}}$.³ If χ is a p -integral character, then we often simply denote its reduction by $\overline{\chi}$.

Next we note the following two facts which are simple, yet very useful for the computation of the reduction of representations of D^\times .

- If σ is a p -integral representation of an open subgroup U of D^\times , $\text{Ind}_U^{D^\times} \sigma$ is also p -integral and the reduction of the representation $\text{Ind}_U^{D^\times} \sigma$ is isomorphic to the semisimplification of the (smooth) induction $\text{Ind}_U^{D^\times} \overline{\sigma}^{\text{ss}}$ of the reduction $\overline{\sigma}^{\text{ss}}$ of σ .
- If σ is a p -integral representation of $E^\times U_D^1$, where E is unramified over F (resp. where E is totally ramified over F), and E is considered as an F -subalgebra of D via an F -embedding, the reduction of σ can be computed by inflating the reduction of the restricted representation

$$\text{Res}_{\mu_E \times \varpi_F^{\mathbb{Z}}}^{E^\times U_D^1} \sigma \quad \left(\text{resp. } \text{Res}_{\mu_F \times \varpi_E^{\mathbb{Z}}}^{E^\times U_D^1} \sigma \right)$$

³Any irreducible representation of D^\times factors through a finite quotient group after a suitable character twist. Therefore we can compute the reduction of irreducible representations by means of Brauer character. For a similar computation with Brauer character, see [BD, Appendice].

to $E^\times U_D^1$, where ϖ_F (resp. ϖ_E) denotes a uniformizer of F (resp. E).
 (Indeed, since $E^\times U_D^1$ is a semidirect product of the normal pro- p -subgroup U_D^1 and a locally profinite subgroup of the form above, every irreducible mod p representation is inflated from an irreducible representation of the latter group. Thus a composition series of a mod p representation of $E^\times U_D^1$ serves as a composition series of the restricted representation.)

The computation of the reduction in the tame case relies on the following key lemmas, which are variants of [BH, 16.2 Lemma].

Let $(E/F, \chi)$ be a minimal pair of positive level $m > 0$. Let us set $n = 2m/e(E|F)$, take $\alpha \in \mathfrak{p}_E^{-m}$ and choose an F -embedding $E \rightarrow D$ exactly as in the previous subsection.

Lemma 3.6. *Suppose that n is even (in which case E/F is necessarily unramified). Let us fix a generator $\zeta_E \in \mu_E$ and a uniformizer ϖ_F of F . Then,*

$$\begin{aligned} \left((\mu_E \times \varpi_F^{\mathbb{Z}}) \backslash E^\times U_D^1 / J_\alpha \right) &\quad \left(\cong \underset{\mu_E}{\sim} \backslash U_D^1 / U_E^1 U_D^{[(n+1)/2]} \right) \\ &= \{J_\alpha\} \amalg \left\{ \prod_{0 \leq i \leq q} \zeta_E^i x J_\alpha \mid x \in U_D^1 \setminus U_E^1 U_D^{[(n+1)/2]} \right\}, \end{aligned}$$

where $\underset{\mu_E}{\sim}$ stands for the equivalence relation induced by the left action of μ_E on $U_D^1 / U_E^1 U_D^{[(n+1)/2]}$ defined by

$$\zeta \cdot x U_E^1 U_D^{[(n+1)/2]} = (\zeta x \zeta^{-1}) U_E^1 U_D^{[(n+1)/2]}$$

for $\zeta \in \mu_E$ and $x \in U_D^1$.

(i.e. if $\zeta \in \mu_E \setminus \mu_F$ and $u \in U_D^1$,

$$\zeta u \zeta^{-1} \equiv u \pmod{U_E^1 U_D^{[(n+1)/2]}} \text{ holds if and only if } u \in U_E^1 U_D^{[(n+1)/2]}.)$$

In particular, we have

$$\# \left((\mu_E \times \varpi_F^{\mathbb{Z}}) \backslash E^\times U_D^1 / J_\alpha \right) = \frac{q^{2[n/4]} - 1}{q + 1} + 1.$$

We only consider a particularly simple situation in the corresponding lemma for ramified extensions.

Lemma 3.7. *Suppose that n is odd (in which case E/F is necessarily totally ramified and $p \neq 2$). Let us fix a uniformizer ϖ_E of E^\times such that $\varpi_E^2 \in F^\times$. Then,*

$$\begin{aligned} (\mu_F \times \varpi_E^{\mathbb{Z}}) \backslash E^\times U_D^1 / J_\alpha &\quad \left(\cong \underset{\varpi_E^{\mathbb{Z}}}{\sim} \backslash U_D^1 / U_E^1 U_D^{[(n+1)/2]} \right) \\ &= \{J_\alpha\} \amalg \left\{ \prod_{0 \leq i \leq 1} \varpi_E^i x J_\alpha \mid x \in U_D^1 \setminus U_E^1 U_D^{[(n+1)/2]} \right\}, \end{aligned}$$

where $\underset{\varpi_E^{\mathbb{Z}}}{\sim}$ stands for the equivalence relation induced by the left action of $\varpi_E^{\mathbb{Z}}$ on $U_D^1 / U_E^1 U_D^{[(n+1)/2]}$ defined by

$$\varpi_E^i \cdot x U_E^1 U_D^{[(n+1)/2]} = (\varpi_E^i x \varpi_E^{-i}) U_E^1 U_D^{[(n+1)/2]}$$

for $i \in \mathbb{Z}$ and $x \in U_D^1$.

(i.e. if $u \in U_D^1$,

$$\varpi_E u \varpi_E^{-1} \equiv u \pmod{U_E^1 U_D^{[(n+1)/2]}} \text{ holds if and only if } u \in U_E^1 U_D^{[(n+1)/2]}.)$$

In particular, we have

$$\# \left((\mu_F \times \varpi_E^{\mathbb{Z}}) \backslash E^\times U_D^1 / J_\alpha \right) = \frac{q^{(n-1)/2} - 1}{2} + 1.$$

Admitting these lemmas for a moment, we compute the reduction first.

Let $(E/F, \chi)$ be an admissible pair and let us choose an F -embedding $E \rightarrow D$. Let us take a decomposition $\chi = (\Phi \circ N_{E/F}) \otimes \chi'$ with Φ a character of F^\times and $(E/F, \chi)$ a minimal pair, denote by m the level of χ' and set $n = 2m/e(E/F)$.

Theorem 3.8. *Let the notation be as above.*

The irreducible admissible representation Π_χ is p -integral if and only if χ is p -integral.

Moreover, assume that χ is a p -integral character.

1. *Assume that n is even (in which case E/F is necessarily unramified). Then we have*

$$\overline{\Pi}_\chi^{\text{ss}} \cong \begin{cases} \left(\left(\pi_{\overline{\chi}}^{\oplus \frac{q^{n/2}+q}{q+1}} \oplus \left(\bigoplus_{\tau \in X} \pi_{\overline{\chi} \otimes \tau}^{\oplus \frac{q^{n/2}-1}{q+1}} \right) \right)^{\text{ss}} & \text{if } n \equiv 0 \pmod{4}, \\ \left(\left(\pi_{\overline{\chi}}^{\oplus \frac{q^{n/2}-q}{q+1}} \oplus \left(\bigoplus_{\tau \in X} \pi_{\overline{\chi} \otimes \tau}^{\oplus \frac{q^{n/2}+1}{q+1}} \right) \right)^{\text{ss}} & \text{if } n \equiv 2 \pmod{4}, \end{cases}$$

where X is the set of non-trivial mod p characters of E^\times which are trivial on $F^\times U_E^1$ (consisting of q elements).

Among mod p representations of the form $\pi_{\bar{\chi}}$ or $\pi_{\bar{\chi} \otimes \tau}$ for some $\tau \in X$, there exists a reducible representation if and only if $\bar{\chi}(-1) = 1$.

2. Assume that n is odd (in which case E/F is necessarily totally ramified and $p \neq 2$). Let us denote by $\bar{\chi}^\dagger$ the extension of $\bar{\chi}$ to $E^\times U_D^1$ such that $\bar{\chi}^\dagger|_{U_D^1} = 1$ as in Proposition 2.7. Then we have

$$\begin{aligned} \bar{\Pi}_\chi^{\text{ss}} &\cong \left(\text{Ind}_{E^\times U_D^1}^{D^\times} \bar{\chi}^\dagger \right)^{\oplus \frac{q^{(n-1)/2}+1}{2}} \oplus \left(\text{Ind}_{E^\times U_D^1}^{D^\times} (\bar{\chi}^\dagger \otimes \iota) \right)^{\oplus \frac{q^{(n-1)/2}-1}{2}} \\ &\cong \left(\bigoplus_{\pi \in I_1} \pi^{\oplus q^{(n-1)/2}} \right) \oplus \left(\bigoplus_{\tilde{\chi} \in I_2} \tilde{\chi}^{\oplus \frac{q^{(n-1)/2}+1}{2}} \right) \oplus \left(\bigoplus_{\tilde{\chi} \in I_2} (\tilde{\chi} \otimes \tilde{\iota})^{\oplus \frac{q^{(n-1)/2}-1}{2}} \right), \end{aligned}$$

where ι (resp. $\tilde{\iota}$) denotes the mod p character of $E^\times U_D^1$ (resp. D^\times) trivial on $U_E U_D^1$ (resp. U_D) sending any uniformizer ϖ_E of E^\times to -1 , I_1 is the set of (isomorphism classes of) two-dimensional irreducible mod p representations of D^\times with a central character $\bar{\chi}|_{F^\times}$ and I_2 is the set of mod p characters of D^\times extending $\bar{\chi}^\dagger$.

If E_0 denotes an unramified quadratic extension of F , we have

$$I_1 = \{ \pi_{\tilde{\chi}} \mid \tilde{\chi} \text{ is a regular mod } p \text{ character of } E_0^\times \text{ extending } \bar{\chi}|_{F^\times} \},$$

$$I_2 = \{ \varphi \circ \text{Nrd} \mid \varphi \text{ is a mod } p \text{ character of } F^\times \text{ such that } (\varphi \circ \text{Nrd})|_{E^\times} = \bar{\chi} \},$$

and

$$(\#I_1, \#I_2) = \begin{cases} (\frac{q+1}{2}, 0) & \text{if } \chi(-1) = -1 \\ (\frac{q-1}{2}, 2) & \text{if } \chi(-1) = 1. \end{cases}$$

Remark 3.9. In most cases (namely, except for the cases where $n = 0$ and where $n = 1, 2$ and $\bar{\chi}$ is irregular) every irreducible admissible mod p representation of D^\times with a central character $\bar{\chi}|_{F^\times}$ occurs in $\bar{\Pi}_\chi^{\text{ss}}$.

Proof. We are immediately reduced to show the theorem for *minimal* representations (i.e. ones which are parametrized by minimal pairs).

If χ is p -integral, Λ is p -integral and therefore Π_χ is p -integral. Conversely, if Π_χ is p -integral, the central character $\chi|_{F^\times}$ is p -integral and so is χ .

In the proof of this theorem and the next one, we denote by 1_G the trivial mod p character of a locally profinite group G .

First assume that n is even and χ is p -integral.

If $n = 0$, the reduction $\bar{\Lambda}^{\text{ss}}$ of the representation Λ of J is $\bar{\chi}^\dagger$ and the assertion is obvious.

For $n > 0$, the reduction $\overline{\Lambda}^{\text{ss}}$ of Λ is trivial on the normal pro- p -subgroup $U_D^{[(n+1)/2]}$ and it is a direct sum of irreducible mod p representations inflated from representations of an abelian group $J_\alpha/U_D^{(n+1)/2}$. The composition factors can be read off by restricting Λ to $\mu_E F^\times$ (cf. Proposition 3.2):

$$\overline{\Lambda}^{\text{ss}} \cong \begin{cases} \overline{\chi}^\dagger|_{J_\alpha} & \text{if } n \equiv 0 \pmod{4} \\ \bigoplus_{\tau \in X} (\overline{\chi} \otimes \tau)^\dagger|_{J_\alpha} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

So we have

$$\begin{aligned} \overline{(\text{Ind}_{J_\alpha}^{E \times U_D^1} \Lambda)}^{\text{ss}} &\cong (\text{Ind}_{J_\alpha}^{E \times U_D^1} \overline{\Lambda}^{\text{ss}})^{\text{ss}} \\ &\cong \begin{cases} \overline{\chi}^\dagger \otimes (\text{Ind}_{J_\alpha}^{E \times U_D^1} 1_{J_\alpha})^{\text{ss}} & \text{if } n \equiv 0 \pmod{4} \\ (\bigoplus_{\tau \in X} (\overline{\chi} \otimes \tau)^\dagger) \otimes (\text{Ind}_{J_\alpha}^{E \times U_D^1} 1_{J_\alpha})^{\text{ss}} & \text{if } n \equiv 2 \pmod{4}. \end{cases} \end{aligned}$$

Thus, we are reduced to show

$$(\text{Ind}_{J_\alpha}^{E \times U_D^1} 1_{J_\alpha})^{\text{ss}} \cong \begin{cases} 1_{E \times U_D^1} \oplus \left((\text{Ind}_{F \times U_D^1}^{E \times U_D^1} 1_{F \times U_D^1})^{\oplus \frac{q^{n/2}-1}{q+1}} \right)^{\text{ss}} & \text{if } n \equiv 0 \pmod{4} \\ 1_{E \times U_D^1} \oplus \left((\text{Ind}_{F \times U_D^1}^{E \times U_D^1} 1_{F \times U_D^1})^{\oplus \frac{q^{(n-2)/2}-1}{q+1}} \right)^{\text{ss}} & \text{if } n \equiv 2 \pmod{4}, \end{cases}$$

i.e.

$$(\text{Ind}_{J_\alpha}^{E \times U_D^1} 1_{J_\alpha})^{\text{ss}} \cong 1_{E \times U_D^1} \oplus \left((\text{Ind}_{F \times U_D^1}^{E \times U_D^1} 1_{F \times U_D^1})^{\oplus \frac{q^{2[n/2]}-1}{q+1}} \right)^{\text{ss}}.$$

Now, if we take a generator $\zeta_E \in \mu_E$ and a uniformizer ϖ_F of F , Mackey's decomposition [Se1, 7.3], [Vil, I.5.5] together with Lemma 3.6 gives

$$\begin{aligned} \text{Res}_{\mu_E \times \varpi_F^{\mathbb{Z}}}^{E \times U_D^1} \text{Ind}_{J_\alpha}^{E \times U_D^1} 1_{J_\alpha} &\cong \bigoplus_{\substack{(\mu_E \times \varpi_F^{\mathbb{Z}})xJ_\alpha \\ \in (\mu_E \times \varpi_F^{\mathbb{Z}}) \setminus E \times U_D^1 / J_\alpha}} \text{Ind}_{J_\alpha^{(x)}}^{\mu_E \times \varpi_F^{\mathbb{Z}}} 1_{J_\alpha^{(x)}} \\ &\cong 1_{\mu_E \times \varpi_F^{\mathbb{Z}}} \oplus \left(\bigoplus_{xU_E^1 U_D^{[(n+1)/2]}} \text{Ind}_{J_\alpha^{(x)}}^{\mu_E \times \varpi_F^{\mathbb{Z}}} 1_{J_\alpha^{(x)}} \right), \end{aligned}$$

where $xU_E^1 U_D^{[(n+1)/2]}$ runs through $(U_D^1/U_E^1 U_D^{[(n+1)/2]}) \setminus \{U_E^1 U_D^{[(n+1)/2]}\}$ and $J_\alpha^{(x)}$ denotes $(\mu_E \times \varpi_F^{\mathbb{Z}}) \cap xJ_\alpha x^{-1}$ for $x \in E \times U_D^1$.

Here we have $J_\alpha^{(x)} = \mu_F \times \varpi_F^{\mathbb{Z}}$ for any $x \in U_D^1 \setminus U_E^1 U_D^{[(n+1)/2]}$ and $n > 2$. Indeed, the index of $J_\alpha^{(x)}$ in $\mu_E \times \varpi_F^{\mathbb{Z}}$ is equal to the length of the μ_E -orbit

containing $xU_E^1U_D^{[(n+1)/2]}$, namely, to $q+1$. The subset $J_\alpha^{(x)}$ surely contains $\mu_F \times \varpi_F^{\mathbb{Z}}$ and the desired equality follows.⁴

Hence, inflating the representation from $\mu_E \times \varpi_F^{\mathbb{Z}}$ to $E^\times U_D^1$, we obtain

$$\left(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} 1_{J_\alpha}\right)^{\mathrm{ss}} \cong 1_{E^\times U_D^1} \oplus \left(\left(\mathrm{Ind}_{F^\times U_D^1}^{E^\times U_D^1} 1_{F^\times U_D^1}\right)^{\oplus \frac{q^{[n/4]}-1}{q+1}}\right)^{\mathrm{ss}}$$

as required. Thus we have the required decomposition of $\overline{\Pi}_\chi$. The assertion about the existence of reducible representations of the form $\pi_{\overline{\chi}}$ or $\pi_{\overline{\chi} \otimes \tau}$ can be verified in much the same way as the proof of similar assertions in Proposition 2.7. This completes the proof of 1.

Next assume that n is odd and χ is p -integral. The strategy for the computation is exactly the same as that in the case of unramified representations.

The reduction $\overline{\Lambda}^{\mathrm{ss}}$ of Λ is isomorphic to $\overline{\chi}^\dagger|_{J_\alpha}$ and the reduction $\overline{(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} \Lambda)}^{\mathrm{ss}}$ of $\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} \Lambda$ is given by

$$\begin{aligned} \overline{(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} \Lambda)}^{\mathrm{ss}} &\cong \left(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} \overline{\Lambda}^{\mathrm{ss}}\right)^{\mathrm{ss}} \\ &\cong \overline{\chi}^\dagger \otimes \left(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} 1_{J_\alpha}\right)^{\mathrm{ss}}, \end{aligned}$$

Then Mackey's decomposition [Se1, 7.3], [Vi1, I.5.5] and Lemma 3.7 yield in turn

$$\left(\mathrm{Ind}_{J_\alpha}^{E^\times U_D^1} 1_{J_\alpha}\right)^{\mathrm{ss}} \cong 1_{E^\times U_D^1} \oplus \left(\mathrm{Ind}_{F^\times U_D^1}^{E^\times U_D^1} 1_{F^\times U_D^1}\right)^{\oplus \frac{q^{(n-1)/2}-1}{2}}.$$

The remaining assertions follow from Proposition 2.7. \square

The theorem for the rest of representations is simpler.

Theorem 3.10. *Assume that $p = 2$. Let $\Pi = \mathrm{Ind}_{E^\times U_D^{(n+1)/2}}^{D^\times} \chi$ represent an element in $\mathcal{A}_1^0(D) \setminus \mathcal{A}_1^{\mathrm{nr}}(D)$ (cf. Theorem 3.5).*

The representation Π is p -integral if and only if the central character Φ of Π is (or equivalently, χ is) p -integral.

Moreover, assume that Π is a p -integral representation. Let us denote by $\overline{\chi}^\dagger$ the extension of $\overline{\chi}$ to $E^\times U_D^1$ such that $\overline{\chi}^\dagger|_{U_D^1} = 1$ as in Proposition 2.7.

⁴Alternatively one can proceed as follows. One inclusion of the equality being trivial, we are to show

$$\text{if } x \in U_D^1 \text{ satisfies } xyx^{-1} \in \mu_E \setminus \mu_F \text{ with } y \in J_\alpha, \text{ then } x \in U_E^1 U_D^{[(n+1)/2]}.$$

Let us put $\zeta = xyx^{-1} \in \mu_E \setminus \mu_F$. Comparing the images under v_D and the canonical projection $U_D \rightarrow U_D/U_D^1$, we see that $yU_D^1 = \zeta U_D^1$. Hence $\zeta x \zeta^{-1} = xy \zeta^{-1} \equiv x \pmod{U_D^1}$ and the claim now follows from Lemma 3.6.

Then we have

$$\begin{aligned}\overline{\Pi}^{\text{ss}} &\cong \left(\text{Ind}_{E^\times U_D^1}^{D^\times} \overline{\chi}^\dagger \right)^{\oplus q^{(n-1)/2}} \\ &\cong \left(\bigoplus_{\pi \in I_1} \pi^{\oplus q^{(n-1)/2}} \right) \oplus \tilde{\chi}^{\oplus q^{(n-1)/2}},\end{aligned}$$

where I_1 is the set of (isomorphism classes of) two-dimensional irreducible mod p representations of D^\times with a central character $\overline{\Phi}(= \overline{\chi}|_{F^\times})$ and $\tilde{\chi}$ is the mod p character of D^\times extending $\overline{\Phi}$ (or equivalently, extending $\overline{\chi}^\dagger$).

Remark 3.11. Under the conditions of the theorem every irreducible admissible mod p representation of D^\times with a central character $\overline{\Phi}$ occurs in $\overline{\Pi}^{\text{ss}}$.

Proof. The proof is parallel to that for the preceding theorem in odd n case.

Assume that $\Pi = \text{Ind}_{E^\times U_D^{(n+1)/2}}^{D^\times} \chi$ is p -integral. Then the reduction $\overline{\left(\text{Ind}_{E^\times U_D^{(n+1)/2}}^{E^\times U_D^1} \chi \right)^{\text{ss}}}$ is isomorphic to $\overline{\chi}^\dagger \otimes \left(\text{Ind}_{E^\times U_D^{(n+1)/2}}^{E^\times U_D^1} 1_{E^\times U_D^{(n+1)/2}} \right)^{\text{ss}}$. Since $\left(\text{Ind}_{E^\times U_D^{(n+1)/2}}^{E^\times U_D^1} 1_{E^\times U_D^{(n+1)/2}} \right)^{\text{ss}}$ is clearly a direct sum of mod p characters trivial on F^\times and an induced representation $\text{Ind}_{E^\times U_D^1}^{D^\times} \nu^\dagger$ only depends on $\nu|_{F^\times}$ if $p = 2$ by Proposition 2.7, the theorem follows easily. \square

We now prove the Lemmas 3.6, 3.7.

proof of Lemma 3.6. Let us set $n = 2m$. We are going to prove the “only if” part of the parenthesized restatement, namely,

if $u \in U_D^1$ satisfies $\zeta u \zeta^{-1} \equiv u \pmod{U_E^1 U_D^m}$ with $\zeta \in \mu_E \setminus \mu_F$, then $u \in U_E^1 U_D^m$,

by induction on m .

As $U_E^1 U_D^1 = U_D^1$, the claim for $m = 1$ is trivial.

Suppose $m \geq 2$ and that the claim holds for $m - 1$. Then it follows that $u \in U_E^1 U_D^{m-1}$ and therefore we may assume $u \in U_D^{m-1}$. Noting that $\min\{h \in \mathbb{Z} \mid 2h \geq m - 1\} = [n/4]$, we see

$$\begin{aligned}U_D^{m-1} \cap (U_E^1 U_D^m) &= U_E^{[n/4]} U_D^m \\ &= \begin{cases} U_E^{m/2} U_D^m & \text{if } n \equiv 0 \pmod{4} \\ U_E^{(m-1)/2} U_D^m & \text{if } n \equiv 2 \pmod{4} \end{cases} \\ &= \begin{cases} U_D^m & \text{if } n \equiv 0 \pmod{4} \\ U_D^{m-1} & \text{if } n \equiv 2 \pmod{4}. \end{cases}\end{aligned}$$

Hence, if m is odd (i.e. $n \equiv 2 \pmod{4}$), we have $u \in U_D^{m-1} \subseteq U_E^1 U_D^m$ and there is nothing to prove.

Suppose, on the other hand, that m is even (i.e. $n \equiv 0 \pmod{4}$). Then we are to show that

if $u \in U_D^{m-1}$ satisfies $\zeta u \zeta^{-1} \equiv u \pmod{U_D^m}$ with $\zeta \in \mu_E \setminus \mu_F$, then $u \in U_D^m$.

Let us take an element ϖ_D of D^\times such that $v_D(\varpi_D) = 1$. Setting $u = 1 + \varpi_D^{m-1}x$ with $x \in \mathfrak{o}_D$ and expressing the claim in additive terms, we obtain the following restatement:

if $x \in \mathfrak{o}_D$ satisfies $\zeta \varpi_D^{m-1} x \zeta^{-1} \equiv \varpi_D^{m-1} x \pmod{\mathfrak{q}^m}$ with $\zeta \in \mu_E \setminus \mu_F$, then $x \in \mathfrak{q}$.

The congruence is equivalent to $\varpi_D^{1-m} \zeta \varpi_D^{m-1} x \zeta^{-1} \equiv x \pmod{\mathfrak{q}}$. If we denote by \bar{x} and $\bar{\zeta}$ the image of x and of ζ in k_D respectively, this means $(\bar{\zeta}^{q-1} - 1)\bar{x} = 0$. Since $\zeta \notin \mu_F$, we surely have $x \in \mathfrak{q}$.

Observing $\#(U_D^1/U_E^1 U_D^{[(n+1)/2]}) = q^{2[n/4]}$, we verify the assertion about the number of double cosets easily. \square

The other lemma is proved in a very similar manner.

proof of Lemma 3.7. Let us set $n = 2m - 1$. We are going to show the claim:

if $u \in U_D^1$ satisfies $\varpi_E u \varpi_E^{-1} \equiv u \pmod{U_E^1 U_D^m}$, then $u \in U_E^1 U_D^m$,

by induction on m .

The claim for $m = 1$ is trivial. Suppose $m \geq 2$ and that the claim holds for $m - 1$. The induction hypothesis shows that $u \in U_E^1 U_D^{m-1}$ and we may therefore assume that $u \in U_D^{m-1}$.

We have $U_D^{m-1} \cap (U_E^1 U_D^m) = U_E^{m-1} U_D^m$ and hence the claim is equivalent to

if $u \in U_D^{m-1}$ satisfies $\varpi_E u \varpi_E^{-1} \equiv u \pmod{U_E^{m-1} U_D^m}$, then $u \in U_E^{m-1} U_D^m$.

If we set $u = 1 + \varpi_E^{m-1}x$ with $x \in \mathfrak{o}_D$ and express the claim in additive terms, it reduces to

if $x \in \mathfrak{o}_D$ satisfies $\varpi_E^m x \varpi_E^{-1} \equiv \varpi_E^{m-1} x \pmod{\mathfrak{p}_E^{m-1} + \mathfrak{q}^m}$, then $x \in \mathfrak{o}_E + \mathfrak{q}$.

Then the congruence is equivalent to $\varpi_E x \varpi_E^{-1} \equiv x \pmod{\mathfrak{o}_E + \mathfrak{q}}$. If we denote by \bar{x} the image of x in k_D , this means

$$\mathrm{tr}_{k_D/k_E}(\bar{x}) - 2\bar{x} = \bar{x}^q - \bar{x} \in k_E,$$

from which it certainly follows that $x \in \mathfrak{o}_E + \mathfrak{q}$, since $p \neq 2$.

In view of the fact that $\#(U_D^1/U_E^1 U_D^{[(n+1)/2]}) = q^{(n-1)/2}$, the assertion about the number of double cosets is not difficult to show. \square

3.3 Reduction of representations of \mathcal{W}_F

In order to compare the reduction of an irreducible admissible representation Π of D^\times and the reduction of the two-dimensional representation R of \mathcal{W}_F which corresponds to Π under LJLC and LLC, we compute the reduction of two-dimensional representations of \mathcal{W}_F in this subsection. We first take up those representations corresponding to tamely ramified representations of D^\times under these correspondences and then the other representations in the case where $p = 2$. Let us denote the set of isomorphism classes of two-dimensional irreducible smooth representations of \mathcal{W}_F over $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ by $\mathcal{G}_2^0(F)$.

Definition 3.12. *We define a subset $\mathcal{G}_2^{\text{nr}}(F)$ of $\mathcal{G}_2^0(F)$ to be the set of isomorphism classes of two-dimensional irreducible representations R of \mathcal{W}_F such that there exists a non-trivial unramified character Φ of F^\times with $(\Phi \circ \mathbf{a}_F) \otimes R \cong R$.*

Theorem 3.13. *1. Let $(E/F, \chi)$ be an admissible pair. Let us set $R_\chi = \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\chi \circ \mathbf{a}_E)$.*

Then the isomorphism class of R_χ only depends on that of $(E/F, \chi)$ and the association $(E/F, \chi) \mapsto R_\chi$ induces a bijection

$$\mathbb{P}_2(F) \simeq \mathcal{G}_2^0(F), \quad \text{if } p \neq 2$$

$$\mathbb{P}_2(F) \simeq \mathcal{G}_2^{\text{nr}}(F), \quad \text{if } p = 2.$$

If $(E/F, \chi)$ is an admissible pair, we have

(a) $\det R_\chi = (\chi|_{F^\times} \otimes \varkappa_{E/F}) \circ \mathbf{a}_F$,
where $\varkappa_{E/F}$ is the character of F^\times associated to the quadratic extension E/F (i.e. the non-trivial character of F^\times trivial on $N_{E/F}(E^\times)$),

(b) $R_{(\Phi \circ N_{E/F}) \otimes \chi} = (\Phi \circ \mathbf{a}_F) \otimes R_\chi$,

(c) R_χ defines a class in $\mathcal{G}_2^{\text{nr}}(F)$ if and only if E/F is unramified.

2. *If $p = 2$, the set $\mathcal{G}_2^{\text{nr}}(F)$ is the image of tamely ramified representations of D^\times under the composite of LJLC and LLC.*

Proof. For 1, see [BH, 34.1 Theorem]. For 2, see [BH, 34.4 Tame Langlands Correspondence, §56]. \square

We now compute the reduction of irreducible representations of \mathcal{W}_F .

Proposition 3.14. *Let $(E/F, \chi)$ be an admissible pair.*

1. *The irreducible representation R_χ is p -integral if and only if χ is p -integral.*

2. Suppose that χ is p -integral. Then we have $\overline{R}_\chi^{\text{ss}} \cong \rho_\chi^{\text{ss}}$.

More explicitly, the reduction is described as follows.

(a) Suppose that E/F is unramified. Let us take ζ_E, ζ_F and ϖ_F as in Remark 2.4.3a.

The reduction $\overline{\chi}$ is regular if and only if $\chi(\zeta_E^{q-1}) \neq 1$.

If $\overline{\chi}$ is irregular, let φ_1 and φ_2 be mod p characters of F^\times such that

$$\varphi_1(\zeta_F) = \overline{\chi}(\zeta_E), \quad \varphi_1(\varpi_F)^2 = \overline{\chi}(\varpi_F) \text{ and } \varphi_2 = \varphi_1 \otimes \overline{\kappa_{E/F}},$$

where $\kappa_{E/F}$ is the character associated to the quadratic extension E/F .

If $\overline{\chi}$ is regular, let us set $\xi = \overline{\chi}$.

(b) Suppose that E/F is (tamely) totally ramified. Let E_0/F be an unramified quadratic extension. Let us take $\zeta_{E_0}, \zeta_F, \varpi_E$ and ϖ_F as in Remark 2.4.3a and 2.4.3b.

The reduction $\overline{\chi}$ is regular if and only if $\chi(-1) \neq 1$.

If $\overline{\chi}$ is irregular, let φ_1 and φ_2 be mod p characters of F^\times such that

$$\varphi_1(\zeta_F)^2 = \overline{\chi}(\zeta_F), \quad \varphi_1(-\varpi_F) = \overline{\chi}(\varpi_E) \text{ and } \varphi_2 = \varphi_1 \otimes \overline{\kappa_{E/F}},$$

where $\kappa_{E/F}$ is the character associated to the quadratic extension E/F .

If $\overline{\chi}$ is regular, let ξ be a mod p character of E_0^\times such that

$$\xi(\zeta_{E_0})^2 = \overline{\chi}(\zeta_F) \text{ and } \xi(\varpi_F) = \overline{\chi}(-1)^{(q+1)/2} \overline{\chi}(\varpi_F).$$

The mod p character ξ is regular.

Then we have

$$\begin{aligned} \overline{R}_\chi^{\text{ss}} &\cong (\rho_{\overline{\chi}})^{\text{ss}} \\ &\cong \begin{cases} (\varphi_1 \circ \mathbf{a}_F) \oplus (\varphi_2 \circ \mathbf{a}_F) & \text{if } \overline{\chi} \text{ is irregular,} \\ \rho_\xi & \text{if } \overline{\chi} \text{ is regular.} \end{cases} \end{aligned}$$

Proof. Applying Proposition 2.9 and Remark 2.10, we obtain the result without much difficulty. \square

Next we compute the reduction of the remaining representations. We are going to use some facts about two-dimensional primitive representations of \mathcal{W}_F as summarized in [BH, §41, §42].

Assume that $p = 2$ and let R represent an element in $\mathcal{G}_2^0(F) \setminus \mathcal{G}_2^{\text{nr}}(F)$.

- Let us denote the group of characters Φ of F^\times such that $(\Phi \circ \mathbf{a}_F) \otimes R \cong R$ by $\mathfrak{T}(R)$. Then the order of $\mathfrak{T}(R)$ is either 1, 2 or 4. There exists a non-trivial (order-two) character $\varkappa \in \mathfrak{T}(R)$ if and only if R is induced from a character of \mathcal{W}_E , where E/F is the quadratic extension associated to the character \varkappa . Thus R is said to be simply imprimitive (resp. triply imprimitive) if $\mathfrak{T}(R)$ consists of two (resp. four) elements whereas R is said to be primitive if $\mathfrak{T}(R)$ is trivial.
- Assume that R is primitive. Then up to isomorphism there exists a unique cubic extension K/F such that $R_K = \text{Res}_{\mathcal{W}_K}^{\mathcal{W}_F} R$ is imprimitive. Let L be the normal closure of K/F . Then $R_L = \text{Res}_{\mathcal{W}_L}^{\mathcal{W}_F} R$ is triply imprimitive. Let M_i/L ($i = 1, 2, 3$) denote the three quadratic extensions such that $\varkappa_{M_i/L} \in \mathfrak{T}(R_L)$ and M be the composite field of M_i/F ($i = 1, 2, 3$). Then M is the normal closure of M_1/F .

If K/F is Galois, then $\text{Gal}(M/F)$ is isomorphic to the alternating group A_4 and R is said to be tetrahedral.

If K/F is not Galois, then R_K is simply imprimitive, $\text{Gal}(M/F)$ is isomorphic to the symmetric group S_4 and R is said to be octahedral. Let E/F be the maximal unramified subextension of L/F . Then $R_E = \text{Res}_{\mathcal{W}_E}^{\mathcal{W}_F} R$ is tetrahedral with L/E totally ramified.

We shall use the notation in the above remark freely in what follows.

Theorem 3.15. *Assume that $p = 2$ and let R represent an element in $\mathcal{G}_2^0(F) \setminus \mathcal{G}_2^{\text{pr}}(F)$. Take the character Φ of F^\times such that $\det(R) = \Phi \circ \mathbf{a}_F$.*

1. *The irreducible representation R is p -integral if and only if Φ is p -integral.*
2. *Suppose that Φ is p -integral. Let $\overline{\Phi}$ denote the reduction of Φ .*
 - (a) *Suppose that R is imprimitive. Then the reduction \overline{R}^{ss} is the direct sum of two identical mod p characters.*
More precisely, let φ be the mod p character of F^\times such that $\varphi^2 = \overline{\Phi}$. Then we have

$$\overline{R}^{\text{ss}} \cong (\varphi \circ \mathbf{a}_F)^{\oplus 2}.$$

- (b) *Suppose that R is tetrahedral. Then the reduction \overline{R}^{ss} is the direct sum of two distinct mod p characters. The two mod p characters are determined by requiring that they are equal when restricted on \mathcal{W}_K and the ratio defines a non-trivial mod p character of $\text{Gal}(K/F)$.*
More explicitly, let φ be the mod p character of F^\times such that $\varphi^2 = \overline{\Phi}$ and let η be a non-trivial mod p character of $\text{Gal}(K/F)$. Set $\varphi_1 = \varphi\eta$ and $\varphi_2 = \varphi\eta^2$. Then we have

$$\overline{R}^{\text{ss}} \cong (\varphi_1 \circ \mathbf{a}_F) \oplus (\varphi_2 \circ \mathbf{a}_F).$$

(c) Suppose that R is octahedral. (Then, $q \equiv -1 \pmod{3}$.) Then the reduction \overline{R}^{ss} is irreducible.

More precisely, let ξ' be the mod p character of E^\times such that $(\xi')^2 = \overline{\Phi} \circ N_{E/F}$ and let η be a non-trivial mod p character of $\text{Gal}(L/E)$. (Taking ζ_E, ζ_F and ϖ_F as in Remark 2.4.3a, we have $\xi'(\zeta_E)^2 = \overline{\Phi}(\zeta_F), \xi'(\varpi_F) = \overline{\Phi}(\varpi_F), \eta(\varpi_E) = 1$ and $\eta(\zeta_E)$ is a primitive third root of unity.) Set $\xi = \xi'\eta$.

Then we have

$$\overline{R}^{\text{ss}} \cong \rho_\xi.$$

Proof. We can easily check the assertion about p -integrality.

First assume that Φ is p -integral and R is imprimitive. If we express R as $R \cong \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \chi$, then we have $\overline{R}^{\text{ss}} \cong \rho_{\overline{\chi}}$. As noted in Remark 2.4, $\overline{\chi}$ is irregular and the assertion follows by Proposition 2.9.

Next assume that Φ is p -integral and R is tetrahedral. As K/F is Galois and the restriction $\left(\text{Res}_{\mathcal{W}_K}^{\mathcal{W}_F} \overline{R}^{\text{ss}}\right)^{\text{ss}} \cong \overline{R}_K^{\text{ss}}$ is the direct sum of two mod p characters by the previous case, we can show by Frobenius reciprocity that the reduction \overline{R}^{ss} is the direct sum of two mod p characters. On the other hand, we readily verify that $R_M = \text{Res}_{\mathcal{W}_M}^{\mathcal{W}_F} R$ is the direct sum of two identical characters. Take an element $x \in \mathcal{W}_F \setminus \mathcal{W}_K$. Since x^3 belongs to \mathcal{W}_M and thus the operator $R(x)^3$ is a scalar, the two eigenvalues of $R(x)$ only differ by a third root of unity. If $R(x)$ were also a scalar, a simple calculation relying on the fact that $\text{Gal}(M/F)$ is isomorphic to the symmetric group S_4 would show that R itself would be the direct sum of characters, which contradicts the irreducibility. Therefore \overline{R}^{ss} is the direct sum of two distinct mod p characters.

Finally assume that Φ is p -integral and R is octahedral. Then the previous two cases determine $\left(\text{Res}_{\mathcal{W}_K}^{\mathcal{W}_F} \overline{R}^{\text{ss}}\right)^{\text{ss}}$ and $\left(\text{Res}_{\mathcal{W}_E}^{\mathcal{W}_F} \overline{R}^{\text{ss}}\right)^{\text{ss}}$, which together determine \overline{R}^{ss} .

The explicit description of the reduction in each cases is seen by simple computations. \square

4 Compatibility with the local Langlands and Jacquet-Langlands correspondence

4.1 Review of tame correspondences

By Theorems 3.13 and 3.4, a certain subset of $\mathcal{G}_2^0(F)$ (namely, $\mathcal{G}_2^0(F)$ if $p \neq 2$ and $\mathcal{G}_2^{nr}(F)$ if $p = 2$) and a certain subset of $\mathcal{A}_1^0(D)$ (namely, $\mathcal{A}_1^0(D)$ if $p \neq 2$ and $\mathcal{A}_1^{nr}(D)$ if $p = 2$) are parametrized by the same set $\mathbb{P}_2(F)$ and therefore there exists a natural bijection between them. However, this is not (the restriction of) the composite of LLC and LJLC. We need to introduce a permutation of $\mathbb{P}_2(F)$ to obtain the right correspondence.

Theorem 4.1. *For any admissible pair $(E/F, \chi)$, there exists a canonical tamely ramified character Δ_χ of E^\times (i.e. Δ_χ is trivial on U_E^1) such that the association $R_\chi \mapsto \Pi_{\Delta_\chi \otimes \chi}$ induces the composite of LLC and LJLC.*

If E/F is unramified, Δ_χ is the unramified character sending any uniformizer to -1 .

If E/F is (tamely) totally ramified, we have

$$\Delta_\chi|_{U_E^1} = 1, \Delta_\chi|_{F^\times} = \varkappa_{E/F},$$

where $\varkappa_{E/F}$ denotes the character of F^\times associated to the quadratic extension E/F .

(The definition of $\Delta_\chi(\varpi_E)$ for a uniformizer ϖ_E is complicated and we omit the details.)

In any case, Δ_χ has order four (if E/F is totally ramified and $q \equiv 3 \pmod{4}$) or order two (otherwise). In particular, if χ is p -integral, then $\Delta_\chi \otimes \chi$ is p -integral.

Proof. See [BH, §34, §56]. Note, however, that if E/F is totally ramified $\Delta_\chi(\varpi_E)$ in [BH] and $\Delta_\chi(\varpi_E)$ here is slightly different (in fact, they are equal up to sign) due to LJLC. \square

4.2 Compatibility in level zero

In the following two subsections we are going to compare the reduction of a representation R of \mathcal{W}_F and the reduction of the representation Π of D^\times which corresponds to R under LLC and LJLC. Our hope for a possible connection with the mod p correspondence failed (at least in a straightforward way) in most cases.

We begin with a positive result. Note that this is due to Vignéras [Vi2].

Theorem 4.2. *Let $(E/F, \chi)$ be an admissible pair. Suppose that χ is p -integral and that the level of χ is zero. (In particular, E/F is unramified.)*

Then $\overline{\chi}$ and $\overline{\Delta_\chi \otimes \chi}$ are regular. The mod p representations $\overline{R}_\chi^{\text{ss}} \cong \rho_{\overline{\chi}}$ and $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}} \cong \pi_{\delta \otimes \overline{\chi}}$ are irreducible and correspond under the mod p correspondence..

Proof. Since $\overline{\Delta_\chi} = \delta$, this follows from Proposition 3.14 and Theorem 3.8. \square

4.3 Incompatibility in higher level

Except for the level zero case, the reduction of a representation Π of D^\times is much more involved than that of its correspondent. We present the following theorem for easy reference although it is lengthy and complicated. We conclude the paper with an observation concerning this intricate result.

Theorem 4.3. *Let $(E/F, \chi)$ be an admissible pair with χ p -integral. Suppose that the level m of χ is positive and set $n = 2m/e(E/F)$.*

1. (cf. Proposition 3.14, Theorem 3.8, Remark 2.6)

Suppose that n is even. (Then E/F is unramified.) Let us set $E_0 = E$. Let us take ζ_{E_0} , ζ_F and ϖ_F as in Remark 2.4.3a.

The mod p character $\overline{\Delta_\chi} \otimes \chi$ is regular exactly when $\overline{\chi}$ is regular. They are regular if and only if $\chi(\zeta_{E_0})^{q-1} \neq 1$.

If $\overline{\chi}$ is regular, $\overline{R_\chi}^{\text{ss}} \cong \rho_{\overline{\chi}}$ is irreducible, whereas if $\overline{\chi}$ is irregular, $\overline{R_\chi}^{\text{ss}}$ is a direct sum of two characters.

Whether $\overline{\chi}$ is regular or not, we have

$$\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}} \cong \begin{cases} \left(\left(\pi_{\delta \otimes \overline{\chi}}^{\oplus \frac{q^{n/2}-q}{q+1}} \oplus \left(\bigoplus_{\tau \in X} \pi_{\delta \otimes \overline{\chi} \otimes \tau}^{\oplus \frac{q^{n/2}+1}{q+1}} \right) \right)^{\text{ss}} & \text{if } n \equiv 2 \pmod{4}, \\ \left(\left(\pi_{\delta \otimes \overline{\chi}}^{\oplus \frac{q^{n/2}+q}{q+1}} \oplus \left(\bigoplus_{\tau \in X} \pi_{\delta \otimes \overline{\chi} \otimes \tau}^{\oplus \frac{q^{n/2}-1}{q+1}} \right) \right)^{\text{ss}} & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

with the notation of Theorem 3.8, 1.

More precisely, the following assertions hold true.

(a) *Suppose that $\chi(\zeta_{E_0})^{q-1} \neq 1$.*

Regardless of the parity of $n/2$ (even if $n = 2$), the image $\pi_{\delta \otimes \overline{\chi}}$ of $\rho_{\overline{\chi}}$ under the mod p correspondence occurs in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$. If $n \equiv 2 \pmod{4}$ (resp. If $n \equiv 0 \pmod{4}$), its multiplicity is one less (resp. more) than that of any other two-dimensional irreducible factors.

Any irreducible mod p representations of D^\times with a central character $(\delta \otimes \overline{\chi})|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$. If $p \neq 2$ and $\chi(-1) = -1$, there exist $(q+1)/2$ such representations and they are all two-dimensional. If $p \neq 2$ (resp. If $p = 2$) and $\chi(-1) = 1$, there exist $(q-1)/2$ (resp. $q/2$) two-dimensional ones and four (resp. only one) one-dimensional ones of all such representations.

(b) *Suppose that $\chi(\zeta_{E_0})^{q-1} = 1$.*

The reduction $\overline{R_\chi}^{\text{ss}}$ is isomorphic to $(\varphi_1 \circ \mathbf{a}_F) \oplus (\varphi_2 \circ \mathbf{a}_F)$ with φ_1 and φ_2 satisfying

$$\varphi_1(\zeta_F) = \overline{\chi}(\zeta_{E_0}), \quad \varphi_1(\varpi_F)^2 = \overline{\chi}(\varpi_F) \text{ and } \varphi_2 = \varphi_1 \otimes \overline{\varkappa_{E_0/F}},$$

where $\varkappa_{E_0/F}$ is the character associated to the quadratic extension E_0/F .

The representation $(\pi_{\delta \otimes \overline{\chi}})^{\text{ss}}$ is isomorphic to $(\varphi_3 \circ \text{Nrd}) \oplus (\varphi_4 \circ \text{Nrd})$ with φ_3 and φ_4 satisfying

$$\varphi_3(\zeta_F) = \overline{\chi}(\zeta_{E_0}), \quad \varphi_3(\varpi_F)^2 = -\overline{\chi}(\varpi_F) \quad \text{and} \quad \varphi_4 = \varphi_3 \otimes \overline{\chi}_{E_0/F}.$$

If $p \neq 2$ and $\tau \in X$ satisfies $\tau(\zeta_{E_0}) = -1$, $(\pi_{\delta \otimes \overline{\chi} \otimes \tau})^{\text{ss}}$ is isomorphic to $(\varphi_5 \circ \text{Nrd}) \oplus (\varphi_6 \circ \text{Nrd})$ with φ_5 and φ_6 satisfying

$$\varphi_5(\zeta_F) = -\overline{\chi}(\zeta_{E_0}), \quad \varphi_5(\varpi_F)^2 = -\overline{\chi}(\varpi_F) \quad \text{and} \quad \varphi_6 = \varphi_5 \otimes \overline{\chi}_{E_0/F}.$$

If $n \equiv 2 \pmod{4}$ (resp. $n \equiv 0 \pmod{4}$) and $p \neq 2$, the multiplicity of $(\varphi_3 \circ \text{Nrd})$ and $(\varphi_4 \circ \text{Nrd})$ is one less (resp. more) than that of $(\varphi_5 \circ \text{Nrd})$ and $(\varphi_6 \circ \text{Nrd})$.

If $n \neq 2$, any irreducible mod p representations of D^\times with a central character $(\delta \otimes \overline{\chi})|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$. If $p \neq 2$ (resp. $p = 2$), there exist $(q-1)/2$ (resp. $q/2$) two-dimensional ones and four (resp. only one) one-dimensional ones (described above) of all such representations.

If $n = 2$, $(\varphi_3 \circ \text{Nrd})$ and $(\varphi_4 \circ \text{Nrd})$ do not occur.

2. (cf. Proposition 3.14, Theorem 3.8, Remark 2.8)

Suppose that n is odd. (Then E/F is totally ramified and $p \neq 2$.) Let E_0/F be an unramified quadratic extension. Let us take $\zeta_{E_0}, \zeta_F, \varpi_E$ and ϖ_F as in Remark 2.4.3a and 2.4.3b.

The mod p character $\overline{\chi}$ is regular if and only if $\chi(-1) = -1$.

If $\overline{\chi}$ is regular, $\overline{R}_\chi^{\text{ss}}$ is irreducible, whereas if $\overline{\chi}$ is irregular, $\overline{R}_\chi^{\text{ss}}$ is a direct sum of two characters.

We have

$$\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}} \cong \left(\bigoplus_{\pi \in I_1} \pi^{\oplus q^{(n-1)/2}} \right) \oplus \left(\bigoplus_{\tilde{\chi} \in I_2} \tilde{\chi}^{\oplus \frac{q^{(n-1)/2} + 1}{2}} \right) \oplus \left(\bigoplus_{\tilde{\chi} \in I_2} (\tilde{\chi} \otimes \tilde{\iota})^{\oplus \frac{q^{(n-1)/2} - 1}{2}} \right),$$

with the notation of Theorem 3.8, 2. The set I_2 of one-dimensional extensions of $\overline{\Delta_\chi \otimes \chi}^\dagger$ is empty if and only if

$$\left(\frac{-1}{q} \right) \chi(-1) = -1,$$

where the Jacobi symbol

$$\left(\frac{-1}{q} \right) = (-1)^{(q-1)/2} = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4} \\ -1 & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

More precisely, the following assertions hold true.

(a) Suppose that $\chi(-1) = -1$ and $q \equiv 1 \pmod{4}$.

We have $\overline{R_\chi}^{\text{ss}} \cong \rho_\xi$, where ξ is a regular mod p character of E_0^\times such that

$$\xi(\zeta_{E_0})^2 = \overline{\chi}(\zeta_F), \quad \xi(\varpi_F) = -\overline{\chi}(\varpi_F).$$

The set I_2 is empty and every element $\pi \in I_1$ is isomorphic to π_ν with ν satisfying

$$\nu(\zeta_{E_0})^{q+1} = -\overline{\chi}(\zeta_F), \quad \nu(\varpi_F) = \overline{\chi}(\varpi_F).$$

The image $\pi_{\delta \otimes \xi}$ of ρ_ξ under the mod p correspondence is in I_2 .

Any irreducible mod p representations of D^\times with a central character $(\overline{\Delta_\chi} \otimes \chi)|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$ with equal multiplicities. There are $(q+1)/2$ such representations.

(b) Suppose that $\chi(-1) = -1$ and $q \equiv 3 \pmod{4}$.

We have $\overline{R_\chi}^{\text{ss}} \cong \rho_\xi$, where ξ is a regular mod p character of E_0^\times such that

$$\xi(\zeta_{E_0})^2 = \overline{\chi}(\zeta_F), \quad \xi(\varpi_F) = \overline{\chi}(\varpi_F).$$

The set I_2 has two elements and every element $\tilde{\chi} \in I_2$ is isomorphic to $(\varphi \circ \text{Nrd})$ with φ satisfying

$$\varphi(\zeta_F)^2 = -\overline{\chi}(\zeta_F), \quad \varphi(\varpi_F) = \varphi(\zeta_F)^{(q-1)/2} \overline{\Delta_\chi}(\varpi_E) \overline{\chi}(\varpi_E).$$

The set I_1 has $(q-1)/2$ elements and every element $\pi \in I_1$ is isomorphic to π_ν with ν satisfying

$$\nu(\zeta_{E_0})^{q+1} = -\overline{\chi}(\zeta_F), \quad \nu(\zeta_{E_0})^2 \neq -\overline{\chi}(\zeta_F) \text{ and } \nu(\varpi_F) = -\overline{\chi}(\varpi_F).$$

The image $\pi_{\delta \otimes \xi}$ of ρ_ξ under the mod p correspondence is in I_2 .

If $n \neq 1$, any irreducible mod p representations of D^\times with a central character $(\overline{\Delta_\chi} \otimes \chi^{\text{ss}})|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$. If $n = 1$, one-dimensional mod p representations of the form $\tilde{\xi} \otimes \tilde{\iota}$ with $\tilde{\xi} \in I_2$ do not occur.

(c) Suppose that $\chi(-1) = 1$ and $q \equiv 1 \pmod{4}$.

The reduction $\overline{R_\chi}^{\text{ss}}$ is isomorphic to $(\varphi_1 \circ \mathbf{a}_F) \oplus (\varphi_2 \circ \mathbf{a}_F)$ with φ_1 and φ_2 satisfying

$$\varphi_1(\zeta_F)^2 = \overline{\chi}(\zeta_F), \quad \varphi_1(\varpi_F) = \overline{\chi}(\zeta_F)^{(q-1)/4} \overline{\chi}(\varpi_E) \text{ and } \varphi_2 = \varphi_1 \otimes \overline{\chi_{E/F}},$$

where $\chi_{E/F}$ is the character associated to the quadratic extension E/F .

The set I_2 has two elements and every element $\tilde{\chi} \in I_2$ is isomorphic to $(\varphi \circ \text{Nrd})$ with φ satisfying

$$\varphi(\zeta_F)^2 = -\overline{\chi}(\zeta_F), \quad \varphi(\varpi_F) = (-\overline{\chi}(\zeta_F))^{(q-1)/4} \overline{\Delta_\chi}(\varpi_E) \overline{\chi}(\varpi_E).$$

The set I_1 has $(q-1)/2$ elements and every element $\pi \in I_1$ is isomorphic to π_ν with ν satisfying

$$\nu(\zeta_{E_0})^{q+1} = -\overline{\chi}(\zeta_F), \quad \nu(\zeta_{E_0})^2 \neq -\overline{\chi}(\zeta_F) \quad \text{and} \quad \nu(\varpi_F) = \overline{\chi}(\varpi_F).$$

If $n \neq 1$, any irreducible mod p representations of D^\times with a central character $(\overline{\Delta_\chi} \otimes \chi)|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$. If $n = 1$, one-dimensional mod p representations of the form $\tilde{\xi} \otimes \tilde{\iota}$ with $\tilde{\xi} \in I_2$ do not occur.

(d) Suppose that $\chi(-1) = 1$ and $q \equiv 3 \pmod{4}$.

The reduction $\overline{R_\chi}^{\text{ss}}$ is isomorphic to $(\varphi_1 \circ \mathbf{a}_F) \oplus (\varphi_2 \circ \mathbf{a}_F)$ with φ_1 and φ_2 satisfying

$$\varphi_1(\zeta_F)^2 = \overline{\chi}(\zeta_F), \quad \varphi_1(\varpi_F) = \varphi_1(\zeta_F)^{(q-1)/2} \overline{\chi}(\varpi_F) \quad \text{and} \quad \varphi_2 = \varphi_1 \otimes \overline{\chi_{E/F}},$$

where $\chi_{E/F}$ is the character associated to the quadratic extension E/F .

The set I_2 is empty and every element $\pi \in I_1$ is isomorphic to π_ν with ν satisfying

$$\nu(\zeta_{E_0})^{q+1} = -\overline{\chi}(\zeta_F), \quad \nu(\varpi_F) = -\overline{\chi}(\varpi_F).$$

Any irreducible mod p representations of D^\times with a central character $(\overline{\Delta_\chi} \otimes \chi)|_{F^\times}$ occur in $\overline{\Pi}_{\Delta_\chi \otimes \chi}^{\text{ss}}$ with equal multiplicities. There are $(q+1)/2$ such representations.

Remark 4.4. For representations in $\mathcal{G}_2^0(F) \setminus \mathcal{G}_2^{\text{nr}}$ and $\mathcal{A}_1^0(D) \setminus \mathcal{A}_1^{\text{nr}}(D)$, we do not give any theorem similar to Theorem 4.3. To have explicit descriptions, we only need to combine Theorem 3.10 and Theorem 3.15, and the situation is simpler.

Remark 4.5. Let R and Π represent elements in $\mathcal{G}_2^0(F)$ and $\mathcal{A}_1^0(D)$ corresponding to each other under the composite of LLC and LJLC, and suppose that they are p -integral. If the reduction $\overline{\Pi}^{\text{ss}}$ were isotypic (i.e. had only one irreducible factor with a multiplicity), one could ask whether the unique irreducible factor of $\overline{\Pi}^{\text{ss}}$ would be the image of the reduction \overline{R}^{ss} under the mod p correspondence (at least when \overline{R}^{ss} is irreducible). However, as we saw in Remark 3.11, any irreducible mod p representations with the suitable central character occur in the reduction in most cases and there seems to be no obvious meaning of the compatibility then. Indeed, irreducible mod p representations of D^\times are automatically trivial on U_D^1 and irreducible (ordinary) representations of D^\times being trivial on U_D^1 simply means that they are of level zero. Therefore, it is no wonder if “the compatibility” holds only in the level zero case.

Still, if \overline{R}^{ss} is irreducible, then the image π of \overline{R}^{ss} under the mod p correspondence does occur in $\overline{\Pi}^{\text{ss}}$ in every case. We may at least ask if there exists any natural way to single out π among other irreducible factors occurring in $\overline{\Pi}^{\text{ss}}$.

1. If R is expressed as R_χ where $(E/F, \chi)$ is an admissible pair with E/F unramified, then π can be characterized as the unique two-dimensional irreducible factor with the multiplicity distinct from that of any other two-dimensional factors. (Under the assumption that E/F is unramified,) all two-dimensional irreducible factors have the same multiplicities if and only if $\overline{\chi}$ is irregular, in accordance with the condition for $\overline{R}_\chi^{\text{ss}}$ to be reducible. However, the multiplicity of π is one less than that of the other two-dimensional factors if $n \equiv 2 \pmod{4}$ whereas it is one more if $n \equiv 0 \pmod{4}$, which appears somewhat odd.
2. If R is expressed as R_χ where $(E/F, \chi)$ is an admissible pair with E/F totally ramified, then π can be characterized as the unique two-dimensional irreducible factor π_ν such that $\nu(\zeta_{E_0})^{q-1} = -1$, i.e. $(\pi_\nu|_{U_D})^2$ acts as a mod p character. (Under the assumption that E/F is totally ramified,) no two-dimensional irreducible factor has the required property if and only if $\overline{\chi}$ is irregular, in accordance with the condition for $\overline{R}_\chi^{\text{ss}}$ to be reducible.
3. If $p = 2$ and R does not arise from any admissible pair, then π can be characterized as the unique two-dimensional irreducible factor π_ν such that $\nu(\zeta_{E_0})^{q-1}$ is a primitive third root of unity, i.e. $(\pi_\nu|_{U_D})^3$ acts as a mod p character. However, provided that $q \equiv -1 \pmod{3}$, the reduction $\overline{\Pi}^{\text{ss}}$ has an irreducible two-dimensional factor with the required property even if \overline{R}^{ss} is reducible.

We do not know any uniform interpretation of these characterizations.

It seems more difficult to say something about the cases where the reductions \overline{R}^{ss} are reducible.

References

- [Br1] C. Breuil, *Sur quelques représentations modulaires et p -adique de $\text{GL}_2(\mathbb{Q}_p)$ I*, Compositio Math. 138 (2003), 165-188
- [Br2] C. Breuil, *Sur quelques représentations modulaires et p -adique de $\text{GL}_2(\mathbb{Q}_p)$ II*, J. Inst. Math. Jussieu 2 (2003), 23-58
- [Br3] C. Breuil, *The emerging p -adic Langlands programme*, Proceedings of the International Congress of Mathematicians, Hyderabad, India (2010), 203-230

- [BD] C. Breuil, F. Diamond, *Formes modulaires de Hilbert modulo p et valeurs d'extensions entre caractères galoisiens*, to appear in Ann. Scient. de l'E.N.S.
- [BH] C. Bushnell, G. Henniart, *The Local Langlands Conjecture for $GL(2)$* , Grundlehren der mathematischen Wissenschaften, 335. Springer-Verlag, 2006
- [Re] I. Reiner, *Maximal Orders*, London mathematical society monographs new series, vol. 28, Oxford University Press, 2003
- [Se1] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Math., vol. 42, Springer-Verlag, 1977
- [Se2] J.-P. Serre, *Local Fields*, Graduate Texts in Math., vol. 67, Springer-Verlag, 1995
- [Vi1] M.-F. Vignéras, *Représentations l -modulaires d'un groupe réductif p -adique avec $l \neq p$* , Progress in Mathematics, vol. 137 Birkhäuser, 1996
- [Vi2] M.-F. Vignéras, *Correspondance modulaire galois-quaternions pour un corps p -adique*, Journées Arithmétiques d'Ulm, Lecture Notes in Math., vol. 1380, Springer-Verlag (1989), 254-266